Intrinsic reflections and strongly rigid Coxeter groups

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Abstract

It is possible for a group W that is abstractly isomorphic to a Coxeter group to have more than one conjugacy class of Coxeter generating sets, and if S and R are two non-conjugate Coxeter generating sets then it may or may not be the case that some element $s \in S$ is conjugate to an element $r \in R$. In this paper we classify the so-called intrinsic reflections: those elements of W whose conjugacy class intersects nontrivially every Coxeter generating set. In combination with previously known results, this leads us to a classification of Coxeter groups for which all Coxeter generating sets are conjugate.

1. *Introduction*

In this paper we answer (in its weakest interpretation) the question *When is a Coxeter system determined by its Coxeter group?*, which is the title of the paper [**7**] of R. Charney and M. Davis. The publication of [**7**] sparked much activity, the ultimate goal of which is solving the isomorphism problem for Coxeter groups. At present, this is still an open problem. However, based on the results that have been obtained since the publication of [**7**], we are able to completely classify those finitely generated Coxeter groups for which the Coxeter generating set is unique up to conjugacy. The final step in this classification is provided by our characterization of intrinsic reflections in finite rank Coxeter systems, the proof of which occupies the major part of this paper.

Let (W, S) be a Coxeter system. Elements of W of the form $s^w := w^{-1}sw$, where $w \in W$ and $s \in S$, are called the *reflections* of (W, S), and $t \in W$ is called an *intrinsic reflection* of the Coxeter group W if t is a reflection of (W, R) for every $R \subseteq W$ such that (W, R) is a Coxeter system. In this paper we provide a complete solution to the following problem.

Problem. Let (W, S) be a Coxeter system of finite rank and let $s \in S$. Give, in terms of the Coxeter diagram of (W, S), conditions that are necessary and sufficient to ensure that s is an intrinsic reflection of W.

This problem was formulated in [**21**] and solved when s is a right-angled generator of (W, S), that is, when s has the property that, for all $t \in S \setminus \{s\}$, the element st either has infinite order or else has order 2. In the first part of this paper we treat generators s that are not right-angled, and thus accomplish a complete solution of the problem above. This is our Theorem 1.

As already indicated in [**21**], combining the characterization of generators that are intrinsic reflections of W with the results in [**5**] and [**16**] provides a characterization of strongly rigid Coxeter systems in terms of their diagrams. A Coxeter system (W, S) is called *strongly rigid* if every Coxeter generating set R of W is of the form S^w for some $w \in W$. Strongly rigid Coxeter systems were introduced in [**3**], and they form a most interesting class of Coxeter systems. Indeed, in the literature there are many results saying that various natural classes of Coxeter

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groups are strongly rigid ([**4**], [**6**], [**7**]). However, it is fair to say that the decisive step towards a characterization of strongly rigid Coxeter systems was made by P.-E. Caprace and P. Przytycki in [**5**]. In fact, it was already indicated in that paper, that it is—in principle—possible to give such a characterization. Our characterization of generators that are intrinsic reflections provides an efficient tool for making this concrete. As a surprise—at least for us—it turns out that the final statement of the characterization of strongly rigid Coxeter systems is relatively concise.

The proof of Theorem 1 commences in Sections 5 to 8, in which we establish conditions that are necessary for a generator $s \in S$ in a Coxeter system (W, S) to be intrinsic. The arguments used in these sections do not need the assumption that S is finite. It turns out that there is a general condition that must be satisfied if s is to be an intrinsic reflection—see Proposition 5.3 or the Fact in Subsection 2.2 below—and some additional conditions originating from specific phenomena in certain finite Coxeter groups. For instance, the well known blowing up and blowing down procedures for Coxeter generating sets, if available, prevent some reflections from being intrinsic. These derive from the isomorphisms (for odd n) between the Coxeter groups of type $A_1 + I_2(n)$ and $I_2(2n)$ and the Coxeter groups of type $A_1 + D_n$ and C_n. The so-called C_3-neighbours of an odd component of S provide a less familiar phenomenon, derived from the existence of a certain automorphism of the Coxeter group of type C_3. If an odd component has a C_3-neighbour then the reflections in the corresponding conjugacy class are not intrinsic.

This first part of the proof of Theorem 1 is summarized in Section 9, where we also introduce the main tool used in the second part, namely the *finite continuation* of a finite order element of W. This concept, first defined in [**11**], requires (W, S) to have finite rank; hence we must assume finite rank in Theorem 1. The proof of Theorem 1 is completed in Section 12.

In the remainder of the paper we investigate strongly rigid Coxeter systems. Our Theorem 3 below provides a characterization of strongly rigid Coxeter systems in terms of their diagrams. This theorem is achieved in two steps. First we provide such a characterization for the strongly reflection rigid Coxeter systems. This is Theorem 2. Its proof, which is given in Section 15, is a synthesis of the main results in [**5**] and [**16**]. We hasten to point out that the by far most substantial part is provided by the main result of [**5**]. Our contribution to Theorem 2 is the observation that only a very small part of [**16**] is needed to deduce Theorem 2 from [**5**]. This observation led us to introduce the notion of a *flexible generator* of a Coxeter system (motivated by the terminology of [**16**]). It turns out that flexible generators are most useful in our context and finally lead to the unexpectedly concise characterization of strongly reflection rigid Coxeter systems in Theorem 2. They are also convenient for proving and stating Theorem 3, which characterizes strongly rigid Coxeter systems in terms of their Coxeter diagrams. The proof of Theorem 3, completed in Section 16, consists essentially of a synthesis of Theorems 1 and 2.

Finally, in Section 17 we record, for ease of future reference, several strong rigidity results for the most frequently discussed classes of Coxeter groups.

2. *Statement of the main results*

2.1. *Some terminology*

The precise statement of our main results requires some preparation, since it involves some non-standard terminology. We start by fixing our notation for Coxeter systems.

Let (W, S) be a Coxeter system. By [**1**, Chap. IV, §1, no. 8, Corollary 3], no proper subset of S generates W. The cardinality of S is called the *rank* of the system; in our main results we assume that the rank is finite. The order of an element $w \in W$ will be denoted by $\mathrm{o}(w)$.

The *Coxeter diagram* of (W, S) is the simplicial graph

$$\Gamma(W, S) := (S, \{\, \{s, t\} \mid \mathrm{o}(st) \neq 2 \,\})$$

with edges labelled by $\{s,t\} \mapsto \mathrm{o}(st)$ (see [**1**, Chap. IV, §1, no. 9]). We shall also make use of the *odd graph* of (W,S), which is defined as the simplicial graph

$$\Omega(W,S) := (S, \{\, \{s,t\} \mid \mathrm{o}(st) \text{ is finite and odd}\,\})$$

again with edges labelled by $\{s,t\} \mapsto \mathrm{o}(st)$.

By [**14**, Theorem 5.5], if J is a subset of S then $(\langle J\rangle, J)$ is a Coxeter system. We define

$$J^{\perp} := \{\, t \in S \mid \mathrm{o}(st) = 2 \text{ for all } s \in J \,\}$$
$$J^{\infty} := \{\, t \in S \mid \mathrm{o}(st) = \infty \text{ for all } s \in J \,\}$$

and we also write $s^{\perp} := \{s\}^{\perp}$ and $s^{\infty} := \{s\}^{\infty}$ for all $s \in S$.

If $K \subseteq J \subseteq S$ then we call K a *direct factor* of J if $J \subseteq K \cup K^{\perp}$. It is clear that, in this situation, $\langle J\rangle = \langle K\rangle \times \langle J \setminus K\rangle$. We call J *irreducible* if $J \neq \emptyset$ and there is no direct factor K of J such that $K \neq \emptyset$ and $K \neq J$. A *component* of J is an irreducible direct factor of J; thus the components of J correspond to the connected components of the Coxeter graph $\Gamma(\langle J\rangle, J)$. We call (W,S) irreducible if S is irreducible.

We write $S^W := \{\, s^w \mid s \in S,\ w \in W \,\}$, the set of reflections of the system (W,S). We let $\ell_S \colon W \to \mathbb{N}$ be the length function associated with the generating set S.

We say that $J \subseteq S$ is *spherical* if $\langle J\rangle$ is a finite group. If J is spherical then $\langle J\rangle$ has a unique element of maximal length; we denote it by ρ_J. A subset J of S is said to be *of* (-1)*-type* if J is spherical and ρ_J is in the centre of $\langle J\rangle$. The group W has nontrivial centre if and only if S has a component of (-1)-type.

The classification of irreducible spherical Coxeter systems is given in [**1**, Chap. VI, §4, no. 1] (for example), where the corresponding Coxeter diagrams are listed and named: they are A_n for $n \geqslant 1$, $B_n = C_n$ for $n \geqslant 2$, D_n for $n \geqslant 4$, $I_2(n)$ for $n \geqslant 5$, H_3, H_4, F_4, E_6, E_7 and E_8, where the subscript is the rank. These names are also called the *types* of the corresponding Coxeter systems. On the occasions that we need to number the vertices of these diagrams, we shall follow Bourbaki's numbering. In particular, for type C_n the edge labelled 4 connects vertices $n-1$ and n, and in type D_n vertices $n-1$ and n are both adjacent to vertex $n-2$.

The irreducible spherical Coxeter systems of types C_n (for $n \geqslant 2$), H_3, H_4, F_4, E_7 and E_8 are of (-1)-type, E_6 is not of (-1)-type, D_n and $I_2(n)$ are of (-1)-type if and only if n is even, and A_n is of (-1)-type if and only if $n = 1$.

The type of a reducible spherical Coxeter system is the formal sum of the types of its components. A reducible spherical system is of (-1)-type if and only if all of its components are of (-1)-type.

2.2. *Intrinsic reflections*

The problem of determining which elements of S are intrinsic reflections of W leads naturally to considering the *odd components* of (W,S). These are the vertex sets of the connected components of the odd graph $\Omega(W,S)$; thus, the set of odd components of (W,S) is a partition of S. If $s, t \in S$ and $\mathrm{o}(st)$ is odd then s and t are conjugate in W, since they are conjugate in $\langle s,t\rangle$. If s and t belong to different odd components then they are not conjugate, since for any odd component M there is a homomorphism $W \to \{\pm 1\}$ mapping elements of M to -1 and elements of $S \setminus M$ to 1. Hence the odd components parametrize the conjugacy classes of reflections of (W,S). It is clear that an element $s \in S$ is intrinsic if and only if all the reflections in its conjugacy class are intrinsic.

Let $M \subseteq K \subseteq S$, and suppose that M is irreducible. We define the *M-principal component of* K to be the component that contains M, and call any other component of K an *M-subsidiary component*. If M is an odd component of (W,S) we define the *even closure* of M to be the set $E(M) := S \setminus M^{\infty}$, and we write $C_0(M)$ for the M-principal component of $E(M)$. The following basic observation is Proposition 5.3 in Section 5.

Fact (Mutable odd components). *Let (W, S) be a Coxeter system and let $M \subseteq S$ an odd component of (W, S). If $E(M)$ has an M-subsidiary component that is of (-1)-type then the elements of M are not intrinsic reflections of W.*

We call the odd component M *mutable* if $E(M)$ has an M-subsidiary component of (-1)-type. We are now able to give a precise statement of our characterization of the elements of S that are intrinsic reflections in W. But first a remark is in order.

Remark. An element $s \in S$ is called a *right-angled generator* of (W, S) if $S \setminus \{s\} \subseteq s^{\perp} \cup s^{\infty}$. Thus s is a right-angled generator if and only if it is an isolated vertex in the Coxeter diagram $\Gamma(\langle S \setminus s^{\infty} \rangle, S \setminus s^{\infty})$. If M is the odd component of (W, S) containing s then s is a right-angled generator if and only if $\{s\} = M = C_0(M)$. The main result of [**21**] provides a Coxeter diagram characterization of the right-angled generators of (W, S) that are intrinsic reflections of W. The characterization given in [**21**] is based on the notion of a blowing down pair, as defined in Definition 7.5 below. Assertion (i) in the statement of our Theorem 1 is just a reformulation of the main result in [**21**]. We include it nonetheless for sake of completeness and later reference.

In view of what we have just said, the task for this paper was to classify intrinsic reflections in odd components M such that $|C_0(M)| \geqslant 2$. This situation is subdivided into three cases, corresponding to assertions (ii), (iii) and (iv) in Theorem 1. In each of these cases, s is an intrinsic reflection if and only if all automorphisms of $\langle C_0(M) \rangle$ preserve reflections. In assertion (iv) the definition of a C_3-neighbour of an odd component is needed. As this definition is somewhat involved, and not essential for the further content of this section, we refer the reader to Section 8, and to Definition 8.2 in particular, for the details.

Theorem 1. *Let (W, S) be a Coxeter system of finite rank, and let $s \in S$. Let M be the odd component of (W, S) containing s, and let $J = C_0(M)$. If the odd component M is mutable then s is not an intrinsic reflection of W. If M is not mutable then the following hold:*

- (i) *if $J = \{s\}$ then s is an intrinsic reflection of W if and only if there is no $t \in S$ such that (s, t) is a blowing down pair;*
- (ii) *if $J \neq \{s\}$ and J is of (-1)-type then s is an intrinsic reflection of W if and only if J is of type C_2, H_3, E_7, or $I_2(4k)$ for some $k \geqslant 2$;*
- (iii) *if J is spherical but not of (-1)-type then s is an intrinsic reflection of W if and only if J is not of type A_5;*
- (iv) *if J is not spherical then s is an intrinsic reflection of W if and only if M has no C_3-neighbours.*

2.3. *Strong rigidity*

We first recall the definitions of strongly rigid and strongly reflection rigid Coxeter systems, proper twists of Coxeter systems and twist rigid Coxeter systems.

Definition 2.1. Let (W, S) be a Coxeter system.

- (i) The system (W, S) is called *strongly rigid* if for each Coxeter generating set $R \subseteq W$ there exists an element $w \in W$ such that $R = S^w$.
- (ii) The system (W, S) is called *strongly reflection rigid* if for each Coxeter generating set $R \subseteq S^W$ there exists an element $w \in W$ such that $R = S^w$.
- (iii) A pair (J, K) is called a *twist* of (W, S) if J is an irreducible spherical subset of S and $K \subseteq S \setminus (J \cup J^{\perp})$ is such that $L := S \setminus (J \cup J^{\perp} \cup K) \subseteq K^{\infty}$. The twist is called *proper* if K and L are both nonempty.
- (iv) The system (W, S) is called *twist rigid* if there are no proper twists of (W, S).

Here are two basic observations concerning these definitions.

Fact (Rigidity). *Let (W, S) be a Coxeter system.*

(i) *The system (W, S) is strongly rigid if and only if it is strongly reflection rigid and each $s \in S$ is an intrinsic reflection of W.*

(ii) *If (W, S) is strongly reflection rigid then it is twist rigid.*

The first of these is clear, the second is proved in Proposition 4.6 below.

Our second theorem provides a characterization of strongly reflection rigid Coxeter systems in terms of their diagrams. To facilitate its statement it is useful to make the following definition.

Definition 2.2. Let (W, S) be a Coxeter system, let $s \in S$ and let J be the irreducible component of $S \setminus s^\infty$ containing s. Then we call s a *flexible generator* of (W, S) if J is spherical and $s^\infty \subseteq (J \setminus \{s\})^\perp$. The set J is called the *flexible factor* of s.

If $s \in S$ is a flexible generator then we define its *type* to be the type of its flexible factor. When more precision is required we say that s is a *flexible generator of type* (X, i) if X is the type of the flexible factor and s corresponds to vertex number i in the diagram X. Note that if J is an irreducible spherical component of (W, S) then each $s \in J$ is a flexible generator of (W, S) whose type is the type of $(\langle J \rangle, J)$.

Theorem 2. *Let (W, S) be a twist rigid Coxeter system of finite rank. Then (W, S) is strongly reflection rigid if and only if the following conditions are both satisfied:*

(i) *there is no flexible generator of type H_3 or H_4;*

(ii) *there is no flexible generator of type $I_2(n)$ with $n = 5$ or $n \geqslant 7$.*

Our goal is to characterize strongly rigid Coxeter systems. By the first of the two rigidity observations above, this can be done by combining our two previous theorems. The conditions that ensure that each generator $s \in S$ is an intrinsic reflection of W can be formulated more concisely with the additional assumption of twist rigidity. To simplify the precise statement we use the concept of a flexible pair, which generalizes slightly the flexible generator concept.

Definition 2.3. Let (W, S) and let $s \neq t \in S$. Then (s, t) is called a *flexible pair* if t is a flexible generator of (W, S) and s is a right-angled generator of (W, S) such that $s^\infty = t^\infty$. The *type* of (s, t) is the type of t.

Theorem 3. *Let (W, S) be a Coxeter system of finite rank. Then (W, S) is strongly rigid if and only if the following conditions all hold:*

(TR) *(W, S) is twist rigid;*

(NM) *(W, S) has no mutable odd component;*

(i) *there is no flexible generator of type H_3 or H_4;*

(ii) *there is no flexible generator of type $I_2(n)$ with $n \geqslant 5$;*

(iii) *there is no flexible generator of type (C_n, n) with $n \geqslant 3$;*

(iv) *there is no flexible generator of type $(C_3, 2)$;*

(v) *there is no flexible pair of type (D_n, n) or $(D_n, n-1)$ with n odd and $n \geqslant 5$;*

(vi) *there is no flexible pair of type A_2 or of type $(A_3, 1)$ or $(A_3, 3)$;*

(vii) *there is no irreducible component of type A_5;*

(viii) *(W, S) is not of type F_4, E_8, or D_{2n} with $n \geqslant 2$.*

2.4. *Background information about the main results*

As already pointed out in the introduction our paper answers, at least partially, the question *When is a Coxeter system determined by its Coxeter group?* posed by Charney and Davis as the title of their paper [**7**]. In that paper they prove the first non-trivial result about strongly rigid Coxeter systems, namely, they show that if a Coxeter group W is capable of acting effectively, properly and cocompactly on a contractible manifold then all Coxeter generating sets in W are conjugate. In the terminology developed in [**3**], this means that each such Coxeter group is strongly rigid. Our Theorem 3 provides a classification of strongly rigid Coxeter systems, characterizing them in terms of their diagrams. Thus it provides a definite answer to the above question if "determined" is interpreted as "determined up to inner automorphisms". The alternative interpretation "determined up to automorphisms" corresponds to the problem of classifying all rigid Coxeter systems, and this is still open at present.

In the introduction we also mentioned that the appearance of [**7**] sparked much activity whose ultimate goal is the solution of the (still open) isomorphism problem for Coxeter groups. The proof of Theorem 3 relies on several results produced by this activity, and we give here the final step. Since Theorem 3 is a complete and definitive result it seems appropriate to provide some detailed information about the main ingredients of its proof, and, in particular, the previous results on which the proof relies.

As explained above, we interpret the title of [**7**] as asking for a characterization of strongly rigid Coxeter systems. This concept was introduced in the paper [**3**], which appeared shortly after the publication of [**7**]. Twistings of Coxeter systems were also introduced in [**3**], along with various notions of rigidity for Coxeter systems. The relevance of [**3**] to Theorem 3 is the fact that the definition of a twist rigid Coxeter system is based on the twisting procedure introduced in that paper.

After the appearance of [**3**] and [**7**] there were various contributions to the strong rigidity problem. Most of them established sufficient conditions for strong rigidity of Coxeter systems in terms of their diagrams. Here we confine ourselves to mentioning those that play an essential role in the proofs of the main results of [**5**] and [**16**], and therefore in the proofs of our Theorems 2 and 3 as well. If (W, S) is a non-spherical Coxeter system whose presentation diagram is a chordfree circuit, then W acts effectively, properly and cocompactly on the affine plane or the hyperbolic plane. Therefore, by the main result of [**7**], any such Coxeter system is strongly rigid. Thus we obtain a condition on a diagram of a Coxeter system that ensures strong rigidity. A second such condition is strong 2-sphericity. We call a Coxeter system (W, S) strongly 2-spherical if it is irreducible, non-spherical, and $\mathrm{o}(st)$ is finite for all $s, t \in S$. By a combination of the results in [**4**] and [**11**], any strongly 2-spherical Coxeter system is strongly rigid. The two criteria for strong rigidity just described were originally proved in the given references by quite different methods. Therefore it is worth mentioning the paper [**6**] in which Caprace and Przytycki study a geometric condition for Coxeter groups that they call bipolarity. One outcome of their work is a uniform and more conceptual proof of strong rigidity for both of these classes of Coxeter systems. See Subsection 17.6 for more information.

Our proof of Theorem 2 is based on results in [**5**] and [**16**]. Both of these references make use of the two criteria for strong rigidity described in the previous paragraph. Therefore, our proof of Theorem 2 indirectly relies on [**4**], [**7**] and [**11**] (or on the alternative direct approach in [**6**]). The key ingredient in our proof of Theorem 2 is the main result of [**5**]. The ideas in the proof of the latter are based on methods from geometric group theory that apply in a wider context, and we refer the reader to the excellent introduction of [**5**] for more information. Whereas our proof of Theorem 2 requires the full power of the main result in [**5**], the amount of [**16**] that is needed from is rather limited (although the rigidity results mentioned above are still needed). In fact, one contribution of the present paper is the notion of a flexible generator, which enables us to extract precisely what is needed from [**16**] in our proof of Theorem 2.

Our proof of Theorem 3 is a straightforward synthesis of our Theorems 1 and 2, together with a systematic use of the notion of a flexible generator. We have already discussed the ingredients of Theorem 2, and so it remains to consider Theorem 1. The proof of this is split into two parts, one of which is published in [**21**], the other covering the major part of the present paper. The paper [**21**] uses only basic results from the theory of Coxeter systems and elementary (but less known) techniques from [**17**], [**19**] and [**22**]. This is not true for the second part of the proof of Theorem 1, presented here, because it uses the full power of the main result in [**11**].

3. *Notation and background material*

Let (W, S) be a Coxeter system. An *S-visible subgroup* of W—or just a *visible subgroup*, if S is understood—is a subgroup of the form $\langle J\rangle$ with J a subset of S. Conjugates of visible subgroups are called *parabolic subgroups*. If $J \subseteq S$ then $(\langle J\rangle, J)$ is a Coxeter system, and the Coxeter diagram $\Gamma(\langle J\rangle, J)$ is the full labelled subgraph of $\Gamma(W, S)$ spanned by J.

If $J \subseteq S$ then $\langle J\rangle$ is normal in W only if J is a union of components of S. Clearly W is the internal direct product of the visible subgroups generated by the components of (W, S). So if $t \in W$ is a reflection then there is a unique component J of (W, S) such that $t \in \langle J\rangle$.

Let V be a real vector space with basis $\{\, \alpha_s \mid s \in S \,\}$ in bijective correspondence with S. As shown in [**14**, Section 5.3] (for example), the group W has a faithful representation as a linear group acting on V, preserving the bilinear form B defined by $B(\alpha_s, \alpha_t) = -\cos(\pi/\mathrm{o}(st))$ for all $s, t \in S$. (We call this the *standard geometrical representation* of W relative to S.) If $s \in S$ then the (-1)-eigenspace of s in V is the 1-dimensional space spanned by α_s. It follows readily that if (W, S) is irreducible then any element in the centre of W must act on V as multiplication by a scalar; furthermore, since B is preserved, the scalar must be ± 1. So in this case $Z(W)$ has order at most 2.

FACT. *The centre of W is an elementary abelian 2-group of order 2^k, where k is the number of components of S of (-1)-type.*

The *root system* of (W, S) in V is the set $\Phi = \{\, w\alpha_s \mid w \in W,\ s \in S \,\}$. It is shown in [**14**, Theorem 5.4] that Φ is the disjoint union of $\Phi^+ = \Phi \cap V^+$ and $\Phi^- = \{\, -\alpha \mid \alpha \in \Phi^+ \,\}$, where $V^+ = \{\, \sum_{s\in S} c_s\alpha_s \mid c_s \geqslant 0 \text{ for all } s \in S \,\}$. Moreover, if $w \in W$ then $\ell_S(w)$ is the number of $\alpha \in \Phi^+$ such that $w\alpha \in \Phi^-$. By [**8**, Proposition 4.1], the root system Φ is finite if and only if (W, S) is spherical. It follows that if $z \in W$ acts on V as multiplication by -1 then $\ell_S(z)$ is the cardinality of Φ^+; so (W, S) is necessarily spherical (since $\ell_S(z)$ is finite) and $z = \rho_S$, the element of W of maximal length. So if $J \subseteq S$ is irreducible then $Z(\langle J\rangle)$ is nontrivial if and only if J is of (-1)-type.

Note also that whenever $J \subseteq S$ is spherical the element ρ_J normalizes the set J, whether or not J is of (-1)-type (by [**8**, Proposition 4.1 (iv) and (iv)$'$]).

3.1. *Visible subgroups and parabolic subgroups*

By [**14**, Proposition 1.10], if $J \subseteq S$ then each coset $w\langle J\rangle$ in W contains a unique minimal length element. (In the notation of Section 3 above, w is minimal length in $w\langle J\rangle$ if and only if $w\alpha_s \in \Phi^+$ for all $s \in J$.) Furthermore, if I and J are subsets of S then each $(\langle I\rangle, \langle J\rangle)$ double coset in W contains a unique element of minimal length.

PROPOSITION 3.1 (Tits). *Let I and J be subsets of S. If d is the minimal length element in $\langle I\rangle d\langle J\rangle$ then $\langle I\rangle \cap d\langle J\rangle d^{-1} = \langle K\rangle$, where $K = I \cap dJd^{-1}$.*

Proof. See [**27**, Lemma 2]. □

It is clear that the intersection of any two parabolic subgroups of W is conjugate in W to $\langle I\rangle \cap d\langle J\rangle d^{-1}$ for some I, J and d as in Proposition 3.1, and hence is also parabolic.

Let $J \subset S$ and $s \in S \setminus J$, and let K be the component of $J \cup \{s\}$ containing s. If K is spherical we define $v[s,J] = \rho_K \rho_{K\setminus\{s\}}$. Then $v[s,J]Jv[s,J]^{-1} \subset J \cup \{s\}$, whenever $v[s,J]$ is defined. The components of $v[s,J]Jv[s,J]^{-1}$ are the same as those of J, apart (possibly) from the components contained in K.

LEMMA 3.2. *Let I and J be subsets of S. Then $\langle I\rangle$ and $\langle J\rangle$ are conjugate in W if and only if there is a $w \in W$ such that $wIw^{-1} = J$. Furthermore, for every such w, the minimal length element in wW_I has the form $v[s_1,J_1]v[s_2,J_2]\cdots v[s_n,J_n]$, where $J_n = I$ and $v[s_iJ_i]J_iv[s_iJ_i]^{-1} = J_{i-1}$ for all $i \in \{1,2,\ldots,n\}$. Moreover, $J_0 = J$.*

Lemma 3.2 was proved in [**12**] (for finite Coxeter groups) and in [**8**] (in general).

LEMMA 3.3. *Let J, $K \subseteq S$ and $w \in W$, and suppose that $J^w \subseteq \langle K\rangle$. Then there is an element $v \in \langle K\rangle$ such that $J^{wv} \subseteq K$. In particular $|J| \leqslant |K|$, and if $|J| = |K|$ then $\langle J\rangle^w = \langle K\rangle$.*

Proof. Let d be the minimal length element in the double coset $\langle J\rangle w\langle K\rangle$. Note that $\langle J\rangle w\langle K\rangle = w\langle K\rangle$, since $J^w \subseteq \langle K\rangle$, so that $d = wv$ for some element $v \in \langle K\rangle$. It now follows that $\langle J\rangle = dv^{-1}\langle w^{-1}Jw\rangle vd^{-1} \subseteq dv^{-1}\langle K\rangle vd^{-1} = d\langle K\rangle d^{-1}$, and so $\langle J\rangle \cap d\langle K\rangle d^{-1} = \langle J\rangle$. But $\langle J\rangle \cap d\langle K\rangle d^{-1} = \langle J \cap dKd^{-1}\rangle$, by Proposition 3.1. Hence $J \subseteq dKd^{-1}$, and $J^{wv} = d^{-1}Jd \subseteq K$, as required. The other claims follow trivially. □

Applying Lemma 3.3 in the case $|J| = 1$ we see that if $K \subseteq S$ and $r \in S^W \cap \langle K\rangle$ then r is conjugate in $\langle K\rangle$ to an element of K. Hence we obtain the following result.

PROPOSITION 3.4. *Let $K \subseteq S$ and let $P = \langle K\rangle$ be the corresponding visible subgroup of W. Then the set of reflections of the Coxeter system (P,K) coincides with the set of reflections of (W,S) contained in P.*

The next proposition, which is Exercise 2d in [**1**] (p. 130), is of crucial importance.

PROPOSITION 3.5 (Tits). *If (W,S) is a Coxeter system and H a finite subgroup of W, then H is contained in a finite parabolic subgroup of W.*

Assume now that (W,S) has finite rank. By Lemma 3.3, if P_1 and P_2 are parabolic subgroups with P_1 a proper subgroup of P_2 then the rank of P_1 is strictly less than that of P_2. So the length of any strictly increasing chain of parabolic subgroups is bounded by the rank of W. Hence it follows from Proposition 3.5 that every finite subgroup of W is contained in a maximal finite subgroup of W, and that every maximal finite subgroup of W is parabolic.

It is clearly a consequence of Lemma 3.2 that if $I \subseteq S$ is maximal subject to being spherical then I is not conjugate to any other subset of S. In view of Propositions 3.5 and 3.1, this yields the following result.

LEMMA 3.6 ([**10**, Lemma 11]). *If $I \subseteq S$ is maximal subject to being spherical then $\langle I \rangle$ is a maximal finite subgroup of W.*

Our next result shows that if X is a set of reflections such that $\langle X \rangle$ is finite, then $\langle X \rangle$ is contained in a finite parabolic subgroup of rank less than or equal to the cardinality of X.

PROPOSITION 3.7. *Let (W, S) be an arbitrary Coxeter system and X a subset of S^W such that $\langle X \rangle$ is a finite group. Then there exist $J \subseteq S$ and $w \in W$ such that J is spherical, $|J| \leqslant |X|$, and $X^w \subseteq \langle J \rangle$.*

Proof. Suppose first that (W, S) is spherical, and let V be the vector space on which W acts in its standard geometrical representation. By [**14**, Theorem 1.12 (d)], the parabolic subgroups of W are precisely the pointwise stabilizers of subspaces of V. Now if $J \subseteq S$ then $\{ v \in V \mid sv = v \text{ for all } s \in J \} = \{ v \in V \mid B(v, \alpha_s) = 0 \text{ for all } s \in J \}$, a subspace of dimension $|S| - |J|$ (since the form B is nondegenerate—indeed, positive definite—in the spherical case). So the rank of a parabolic subgroup is the codimension of its fixed point space. Now if $X = \{r_1, r_2, \ldots, r_k\} \subseteq S^W$ then $\mathrm{Fix}_V(X) = \{ v \in V \mid rv = v \text{ for all } r \in X \}$ has codimension at most k (equality occurring if the roots corresponding to the reflections r_i are linearly independent). So the rank of the pointwise stabilizer of $\mathrm{Fix}_V(X)$ is at most k, as required.

For the general case, we apply Proposition 3.5 to deduce that there exist a spherical subset K of S and an element $v \in W$ such that $\langle X \rangle^v \leqslant \langle K \rangle$. But now Proposition 3.4 yields that $X^v \subseteq K^{\langle K \rangle}$, and so, by the spherical case, there exist $J \subseteq K$ and $u \in \langle K \rangle$ such that $\langle X^v \rangle^u \leqslant \langle J \rangle$ and $|J| \leqslant |X^v| = |X|$. Thus the assertion holds with $w = vu$. □

3.2. *Involutions*

We make substantial use of the classification of involutions in Coxeter groups, due to Richardson [**24**]. It is convenient to use the following formulation.

PROPOSITION 3.8 ([**10**, Proposition 5]). *If $r \in W$ is an involution then there exist $w \in W$ and $J \subseteq S$ such that J is of (-1)-type and $r = w^{-1}\rho_J w$ with $\ell_S(r) = \ell_S(\rho_J) + 2\ell_S(w)$.*

DEFINITION 3.9. The *rank* of an involution $r \in W$ is defined to be the dimension of its (-1)-eigenspace in the standard geometrical realization.

If $J \subseteq S$ is of (-1)-type then the (-1)-eigenspace of ρ_J is spanned by $\{ \alpha_s \mid s \in J \}$, and so the rank of ρ_J is $|J|$. Reflections are involutions of rank 1. The rank of an involution is clearly conjugacy invariant, but depends on the Coxeter generating set S.

We shall use the rank in several places when we need to prove that an involution is not a reflection. The three cases in the next lemma correspond to the specific situations that arise.

LEMMA 3.10. *Let (W, S) be a Coxeter system and let $r \in W$ be an involution. Then $r \notin S^W$ if any of the following conditions hold:*

(i) *$r = \rho_J$, where $J \subseteq S$ is of (-1)-type and $|J| \geqslant 2$;*
(ii) *$r = \rho_I \rho_J$ where I, $J \subseteq S$ are nonempty and both of (-1)-type, and $I \subseteq J^{\perp}$;*
(iii) *$r = s\rho_J$ where $J \subseteq S$ is of (-1)-type, $s \in J$, and $|J| \geqslant 3$.*

Proof. In case (i) the rank of r is $|J|$, in case (ii) it is $|I| + |J|$, and in case (iii) it is $|J| - 1$. So in no case is the rank of r equal to 1. □

The next lemma shows that, for a given S, each involution $w \in W$ has a uniquely defined *type*, namely the type of the Coxeter system $(\langle J \rangle, J)$, where J is as in Proposition 3.8.

LEMMA 3.11 ([**11**, Lemma 21]). *Suppose that I and J are subsets of S that are both of (-1)-type. Then $\{ w \in W \mid w\rho_I w^{-1} = \rho_J \} = \{ w \in W \mid w\langle I \rangle w^{-1} = \langle J \rangle \}$.*

We also need the following result.

PROPOSITION 3.12. *Suppose that $I \subseteq S$ is irreducible and of (-1)-type, and has at least two elements. If $J \subseteq S$ and $\langle J \rangle = w\langle I \rangle w^{-1}$ for some $w \in W$, then $J = I$.*

Proof. Suppose, for a contradiction, that there is a $w \in W$ such that $I \neq wIw^{-1} \subseteq S$. By Lemma 3.2 there must be a sequence $J_0, J_1, \ldots, J_k$ of subsets of S such that $J_0 = I$ and $J_k = wIw^{-1}$, and for each $i \in \{1, 2, \ldots, k\}$,

(i) there is an $s_i \in S$ such that $\{s_i\} \cup J_{i-1} = J_i \cup J_{i-1}$,
(ii) $\{s_i\} \cup J_{i-1}$ is spherical,
(iii) $J_i = \sigma_i \rho_{i-1} J_{i-1} \rho_{i-1} \sigma_i$, where $\rho_{i-1} = \rho_{J_{i-1}}$ and $\sigma_i = \rho_{\{s_i\} \cup J_{i-1}}$.

Choosing w so that the k above is minimal (subject to $I \neq wIw^{-1} \subseteq S$) clearly makes $k = 1$. So I is irreducible, of (-1)-type, and of rank at least 2, and there is an $s \in S$ such that $K = I \cup \{s\}$ is spherical, and $\rho_K \rho_I I \rho_I \rho_K \neq I$. We now use a case by case analysis to show that in fact this never happens.

Since $\rho_I I \rho_I = I$, we must have $\rho_K I \rho_K \neq I$, and this certainly means that ρ_K is not central in K. So K is not of (-1)-type. Since I is of (-1)-type, it follows that $\langle K \rangle$ is not the direct product of $\langle I \rangle$ and $\langle s \rangle$. Hence K is irreducible. Furthermore, the rank of K is at least 3, since the rank of I is assumed to be at least 2.

The only irreducible Coxeter systems of rank 3 or more that are not of (-1)-type are the E_6 system, the A_n systems and the D_{2k+1} systems. In E_6 there are three irreducible visible subsystems of rank 5, one of type A_5 and two of type D_5. Since these are not of (-1)-type, they are not candidates for I. Similarly, the irreducible rank $n - 1$ subsystems of A_n are both of type A_{n-1}, and not of (-1)-type since $n \geqslant 3$. So K must be of type D_{2k+1}, with I of type D_{2k} (since the only other irreducible rank $2k$ subsystems are of type A_{2k}, and not of (-1)-type). But the longest element of D_{2k+1} normalizes the visible subgroup of type D_{2k}, swapping two of the generators and centralizing the others, contrary to the requirement that $\rho_K I \rho_K \neq I$. □

4. *Replacement pairs*

In order to prove Theorem 1 we have to show that a generator $s \in S$ of a Coxeter system (W, S) is not intrinsic under certain conditions. Thus we will have to establish the existence of a Coxeter generating set R of W such that $s \notin R^W$. Our strategy for the construction of such a set R is based on a local-to-global principle that we present in this section. This principle is based on the notion of a replacement pair for a Coxeter system. As a first basic example for an application of replacement pairs we discuss the twisting procedure for Coxeter generating sets, which was introduced in [**3**]. We shall need the following two well known results.

PROPOSITION 4.1. *Let W be a group and S, T subsets of W, and suppose that $(\langle S\rangle, S)$ and $(\langle T\rangle, T)$ are Coxeter systems. If W is the direct product of $\langle S\rangle$ and $\langle T\rangle$ then $(W, S\cup T)$ is a Coxeter system, with $\mathrm{o}(st) = 2$ whenever $s\in S$ and $t\in T$. The Coxeter diagram $\Gamma(W, S\cup T)$ is the disjoint union $\Gamma(\langle S\rangle, S) + \Gamma(\langle T\rangle, T)$.*

PROPOSITION 4.2. *Let W be a group and S, T subsets of W, and suppose that $(\langle S\rangle, S)$ and $(\langle T\rangle, T)$ are Coxeter systems. Suppose also that $(\langle S\rangle\cap\langle T\rangle, S\cap T)$ is a Coxeter system and that W is the free product of $\langle S\rangle$ and $\langle T\rangle$ with amalgamation of $\langle S\rangle\cap\langle T\rangle$. Then $(W, S\cup T)$ is a Coxeter system, with $\mathrm{o}(st) = \infty$ whenever $s\in S\setminus(S\cap T)$ and $t\in T\setminus(S\cap T)$. The Coxeter diagram $\Gamma(W, S\cup T)$ is formed from $\Gamma_S = \Gamma(\langle S\rangle, S)$ and $\Gamma_T = \Gamma(\langle T\rangle, T)$ by identifying the subgraphs of Γ_S and Γ_T spanned by $S\cap T$, and adding ∞-labelled edges $\{s, t\}$ for all $s\in S\setminus(S\cap T)$ and $t\in T\setminus(S\cap T)$.*

Proofs of Propositions 4.1 and 4.2 can be found in [**21**, Proposition 3.3].

DEFINITION 4.3. Let (W, S) be a Coxeter system, let K be a subset of S and let K' be a subset of W. We say that (K, K') is a *replacement pair* for (W, S) if $(W, (S\setminus K)\cup K')$ is also a Coxeter system.

Of course, if (K, K') is a replacement pair for (W, S), and $R = (S\setminus K)\cup K'$, then (K', K) is a replacement pair for (W, R).

The usefulness of the replacement pair concept stems from the next lemma, which allows us to unify several different constructions. Recall that if (W, S) is a Coxeter system and $J\subseteq S$ then we define

$$J^{\perp} = \{\, s\in S \mid \mathrm{o}(st) = 2 \text{ for all } t\in J\,\},$$
$$J^{\infty} = \{\, s\in S \mid \mathrm{o}(st) = \infty \text{ for all } t\in J\,\}.$$

LEMMA 4.4. *Let (W, S) be a Coxeter system and $J\subseteq S$, and suppose that (K, K') is a replacement pair for the Coxeter system $(\langle J\rangle, J)$. If $S\setminus(J\cup J^{\perp})\subseteq K^{\infty}$ then (K, K') is a replacement pair for the Coxeter system (W, S).*

Proof. Put $G = \langle J\rangle$ and $E = \langle J\cup J^{\perp}\rangle$, noting that E is the direct product of the Coxeter groups G and $\langle J^{\perp}\rangle$. Define $J' = K'\cup(J\setminus K)$. Since (K, K') is a replacement pair for (G, J) it follows that (G, J') is also a Coxeter system. Hence $(E, J'\cup J^{\perp})$ is a Coxeter system, by Proposition 4.1.

Write $H = S\setminus(J\cup J^{\perp})$ and $L = (J\setminus K)\cup J^{\perp}$. Since $S = (L\cup K)\cup(L\cup H)$ and $\mathrm{o}(st) = \infty$ for all $s\in K$ and $t\in H$, it follows that W is the free product of $\langle L\cup K\rangle$ and $\langle L\cup H\rangle$ with amalgamation of $\langle L\rangle$. Let $D = \langle L\cup H\rangle$, noting that $\langle L\cup K\rangle = \langle J\cup J^{\perp}\rangle = E$, and recall that $J'\cup J^{\perp}$ is a Coxeter generating set for E. The set $X = (J'\cup J^{\perp})\cap(L\cup H)$ generates a subgroup of $D\cap E = \langle L\rangle$, and since $L = (J\setminus K)\cup J^{\perp}\subseteq J'\cup J^{\perp}$, we see that $L\subseteq X$. Hence $\langle X\rangle = \langle L\rangle$. So it follows from Proposition 4.2 that $(J'\cup J^{\perp})\cup(L\cup H) = J'\cup J^{\perp}\cup H = K'\cup(S\setminus K)$ is a Coxeter generating set for W, as required. □

Let (G, M) be a Coxeter system and J, $H\subseteq M$ with J spherical and $H\subseteq J^{\perp}$. Put $K = M\setminus(J\cup H)$. Since $(M\setminus K)\cup\rho_J K\rho_J = J\cup H\cup\rho_J K\rho_J = \rho_J(J\cup H\cup K)\rho_J = \rho_J M\rho_J$, a conjugate of M, it is clear that $(K, \rho_J K\rho_J)$ is a replacement pair for (G, M). Now suppose that (W, S) is a Coxeter system, for which G as above is a visible subgroup, and $S\setminus M\subseteq K^{\infty}$.

Then $S \setminus (M \cup M^{\perp}) \subseteq K^{\infty}$, and thus Lemma 4.4 tells us that $(K, \rho_J K \rho_J)$ is a replacement pair for (W, S). In fact this is just the twisting construction introduced in [**3**].

Proposition 4.5 ([**3**, Theorem 4.5.]). *Let (W, S) be a Coxeter system, and suppose that J, H, K and L are pairwise disjoint subsets of S with $S = J \cup H \cup K \cup L$. Suppose also that J is spherical, $H \subseteq J^{\perp}$ and $L \subseteq K^{\infty}$. Then $(K, \rho_J K \rho_J)$ is a replacement pair for (W, S).*

The next proposition shows that if (W, S) admits a proper twist then it is not strongly rigid.

Proposition 4.6. *Let (W, S) and J, H, K and L be as in Proposition 4.5. If K and L are nonempty and not subsets of $J^{\perp}$, and if J is irreducible, then S and $R := (S \setminus K) \cup \rho_J K \rho_J$ are not conjugate in W.*

Proof. Suppose, for a contradiction, that $w \in W$ and $R^w = S$. Put $M = J \cup H \cup K$ and $N = J \cup H \cup L$, and let $\rho = \rho_J$. Then $(J \cup H)^{\rho} = J \cup H$, and $M^{\rho w} \subseteq R^w \subseteq S$. But there is no $s \in (S \setminus M)$ such that $v[s, M]$ is defined, since $\mathrm{o}(st) = \infty$ for all $s \in L$ and $t \in K$, and both L and K are nonempty. So it follows from Lemma 3.2 that $\rho w \in \langle M \rangle$, and thus $w \in \langle M \rangle$. Similarly, since $N^w \subseteq R^w \subseteq S$ and there is no $s \in (S \setminus N)$ such that $v[s, N]$ is defined, it follows that $w \in \langle N \rangle$. Hence $w \in \langle J \cup H \rangle$, and since $(J \cup H)^w = J \cup H$ it follows that $w = \rho_I$ where I is a union of components of $J \cup H$. If $J \subseteq I$ then $L^w \nsubseteq S$, since $L \nsubseteq J^{\perp}$, and if $J \nsubseteq I$ then $K^{\rho w} \nsubseteq S$, since $K \nsubseteq J^{\perp}$. Both alternatives contradict $R^w = S$. □

The twisting procedure described in Proposition 4.5 gives a way of constructing a new Coxeter generating set, replacing a subset K of S by a suitable conjugate $K' = \rho K \rho$, where ρ is a longest element in some visible spherical subgroup. If the twist is proper then $R := K' \cup (S \setminus K)$ is not conjugate to S in W. However, R and S produce the same set of reflections $R^W = S^W$, since by its definition R is a subset of S^W, and dually $S \subseteq R^W$ (since $(K', \rho K' \rho)$ is a replacement pair for (W, R)).

5. *Intrinsic reflections and odd components*

In this section we introduce intrinsic reflections and provide some basic observations and examples concerning this notion. Whether or not an element r in a Coxeter group W is an intrinsic reflection of W depends on its conjugacy class in W. For a given Coxeter system (W, S), the reflections are the conjugates of elements of S, and we are naturally led to consider the odd components of (W, S), since these are precisely the intersections of S with the various conjugacy classes of reflections. In the final part of this section we obtain a condition that, if satisfied, ensures that a generator $s \in S$ is not an intrinsic reflection of W. This condition describes, in fact, the generic case in which a generator is not an intrinsic reflection of W.

Definition 5.1. Let W be a Coxeter group. An involution r is called an intrinsic reflection of W if $r \in S^W$ for each Coxeter generating set S of W.

We begin with some examples.

(1) Let $n \geqslant 3$ and let (W, S) be a Coxeter system of type $I_2(n)$ (so that W is dihedral of order $2n$, and $|S| = 2$). If n is odd or divisible by 4, then the elements of S are intrinsic reflections of W. If $n = 4k + 2$ for some $k \geqslant 1$ then there are two nonconjugate Coxeter generating sets

of cardinality 3: if $S = \{s,t\}$ and $z = (st)^{2k+1}$ (the central involution), then $\{z, s, tst\}$ and $\{z, t, sts\}$ are Coxeter generating sets. So W has no intrinsic reflections.

(2) If W is elementary abelian group of order 2^n then W is a Coxeter group of type nA_1. If $n \geqslant 2$ then no involution is an intrinsic reflection.

(3) Let (W,S) be irreducible and non-spherical. It was proved in [**11**] that if (W,S) is 2-spherical (meaning that $\mathrm{o}(st)$ is finite for all s, $t \in S$) then all elements of S are intrinsic reflections of W. The proof given there uses the finite continuation, which will also be our main tool when it comes to proving that an element of S is an intrinsic reflection of W.

Proving Theorem 1 involves showing that, under certain conditions, an element $s \in S$ is not an intrinsic reflection of W. The following observation is useful in this regard.

Lemma 5.2. *Let (W,S) be a Coxeter system, and let $M \subseteq S$ be an odd component of S. If there exists $M' \subseteq W$ such that (M,M') is a replacement pair for (W,S) and no element of M' is conjugate to any element of M, then the elements of M are not intrinsic reflections of W.*

Proof. Since (M,M') is a replacement pair for (W,S), the set $R := (S \setminus M) \cup M'$ is a Coxeter generating set for W. Since M is an odd component it has at least one element s, and s is not conjugate to any element of $S \setminus M$. Since we are also given that s is not conjugate to any element of M', it is not conjugate to any element of R. Hence s is not an intrinsic reflection. □

Recall that if M is an odd component of S then its even closure is the set $E(M) = S \setminus M^\infty$. The component of $E(M)$ containing M is called the M-principal component of $E(M)$, and is denoted by $C_0(M)$. The other components of $E(M)$ are called the M-subsidiary components. Clearly $E(M) = C_0(M) \cup C_0(M)^\perp$.

Proposition 5.3. *Let M be an odd component of the Coxeter system (W,S), and suppose that $E(M)$ has a subsidiary component K that is of (-1)-type. Then the elements of M are not intrinsic reflections.*

Proof. Let $J = C_0(M) \cup K$ and $G = \langle J \rangle$, and observe that (G,J) is a Coxeter system with exactly two irreducible components, namely $C_0(M)$ and K. Let $z = \rho_K$ be the nontrivial central element of $\langle K \rangle$, and note that $z \in Z(G)$.

Observe that there is a endomorphism $\alpha \colon G \to G$ such that

$$\alpha(s) = \begin{cases} zs & \text{if } s \in M, \\ s & \text{if } s \in J \setminus M. \end{cases}$$

Furthermore, since $z \in \langle K \rangle \subseteq \langle J \setminus M \rangle$ it follows that $\alpha(z) = z$, and hence α^2 is the identity map on G. Thus α is an automorphism of G, and therefore $(G, \alpha(J))$ is a Coxeter system. Since clearly $\alpha(J) = zM \cup (J \setminus M)$ it follows that (M, zM) is a replacement pair for (G,J). But since $J \cup J^\perp$ includes all the subsidiary components of $E(M)$ and no elements of M^∞, we see that $S \setminus (J \cup J^\perp) = S \setminus E(M) = M^\infty$. Hence Lemma 4.4 applies, and tells us that (M, zM) is a replacement pair for (W,S).

Let $s \in M$. As $z = \rho_K$ and $s = \rho_{\{s\}} \in E(M) \setminus K \subseteq K^\perp$, it follows by assertion (ii) of Lemma 3.10 that $zs \notin S^W$. Hence $zM \cap S^W = \emptyset$, and it follows from Lemma 5.2 that the elements of M are not intrinsic reflections of W. □

6. *Intrinsic reflections in irreducible spherical systems*

Our first proposition in this section is well known, but since there is no convenient reference in the literature we sketch a proof.

PROPOSITION 6.1. *Let W be a finite group and let S and R be two Coxeter generating sets for W such that (W, S) and (W, R) are both irreducible. Then there exists an automorphism of W mapping S onto R. In particular (W, S) and (W, R) are of the same type.*

Proof. This is the same as saying that irreducible spherical Coxeter groups of different types are not isomorphic. There are many places in the literature where the groups are described (see, for example, 2.10–2.13 of [**14**], and Appendix 5 of [**26**]), and there are many group theoretic ways to distinguish them from each other.

One tool that can be used in this task, making use of ideas we already have at hand, involves finding the number of classes of involutions. This is easily done by inspecting the Coxeter diagram to find all $J \subseteq S$ of (-1)-type, and then using Lemma 3.11 and the methods of [**12**] to determine conjugacy. It turns out that two subsets of (-1)-type are conjugate if and only if they have the same type, except that in E_7 there are two classes of involutions of type $3A_1$, and in D_{2k} there are two classes of type kA_1. To make this correct, when S has two odd components we need to colour them differently, and say that subsets are of different types if the colours do not match. Additionally, in type D_n we say that the two element subset of S corresponding to vertices $n-1$ and n of $\Gamma(W, S)$ is of type D_2 rather than $2A_1$.

For every irreducible finite Coxeter group the abelianized group has order 2 or order 4. Order 4 occurs for types C_n and F_4, and for $I_2(n)$ when n is even. These can be distinguished from each other by the number of classes of involutions and by the order. For $I_2(n)$ (with n even) there are three classes of involutions, and W has order $2n$. For C_3 there are five classes of involutions, for F_4 there are seven, for C_4 there are eight, and for C_n with $n > 4$ there are more than 8. (In fact the number is $\lfloor n^2/4 \rfloor + n$.) So there are no isomorphisms between these groups.

Consider now the groups whose abelianization has order 2. Types A_1, A_2 and $I_2(n)$ (for n odd) are distinguished from the others by having only one class of involutions, and from each other by their orders. Types E_6, A_n for $n > 1$ and D_n for n odd and $n > 3$, are distinguished from D_n for n even, H_3, H_4, E_7 and E_8 by having trivial centre.

We refer the reader to [**14**, Section 2.10] for the proof that the Coxeter group of type A_n is isomorphic to the symmetric group on $n+1$ letters, and thus has order $(n+1)!$, while the group of type D_n has order $2^{n-1}n!$. It is then easily checked that no group of type A_n has the same order as any group of type D_m with $m > 3$ and m odd. The group of type E_6 has four classes of involutions, the same as D_5, but E_6 contains D_5 as a proper (visible) subgroup. To distinguish E_6 from A_7 and A_8 (which also have four classes of involutions) one can use their orders, for example.

Types H_3, H_4, E_7 and E_8 have, respectively, 3, 4, 9 and 9 classes of involutions. (Note that the two classes of type $3A_1$ in E_7 are fused in E_8.) In D_4 and D_6 there are, respectively, 6 and 10 classes of involutions, and there are more than 10 in D_n when n is even and $n > 3$. Since E_7 is a proper subgroup of E_8, there are no isomorphisms here. □

LEMMA 6.2. *Suppose that (W, S) is of type $I_2(2n)$ or C_n where n is odd and $n \geqslant 3$. Let M be an odd component of S with $|M| = 1$, and let $s, a \in S$ with $M = \{s\}$ and $\mathrm{o}(as) > 2$. Let $z = \rho_S$, the longest element of W. Then $R := \{z, sas\} \cup (S \setminus \{s\})$ is a Coxeter generating set of W such that s is not in R^W. In particular, $(\{s\}, \{z, sas\})$ is a replacement pair for (W, S). Moreover, s is not an intrinsic reflection of W.*

Proof. The first claim is well known and follows, for instance, from [**21**, Lemma 3.6]. The second follows from Lemma 5.2, since s is not conjugate to either z or sas. □

Lemma 6.3. *Let (W, S) be an irreducible spherical Coxeter system. If W has a nontrivial direct product decomposition then (W, S) is of (-1)-type and is isomorphic to $\langle z\rangle \times G$, where $\langle z\rangle = Z(W)$ has order 2 and G is indecomposable. Furthermore, one of the following holds:*

(i) *(W, S) is of type $I_2(2n)$ or C_n with n odd and $n \geqslant 3$, and G is a Coxeter group of type $I_2(n)$ or D_n (where $D_3 = A_3$),*
(ii) *(W, S) is of type H_3 or E_7 and G is not isomorphic to any Coxeter group.*

Proof. This is [**23**, Proposition 7.2]. Note that for types H_3 and E_7 the abelianization of W has order 2; hence in (ii) the group G is the derived group of W, and is perfect (which shows that it is not a Coxeter group). In fact G is simple, isomorphic to Alt(5) in one case and $\mathrm{O}_6^-(2)$ in the other. See [**14**, p. 45]. □

Lemma 6.4. *Assume that (W, S) is of (-1)-type, with S irreducible and $|S| \geqslant 3$. Let M be an odd component of S, and let $z := \rho_S$ be the central involution of W. Let $\varphi\colon W \to W$ be the endomorphism given by $\varphi(s) = zs$ for $s \in M$ and $\varphi(s) = s$ for $s \in S \setminus M$. Then φ is an automorphism, unless (W, S) is of type H_3 or E_7, or $|M| = 1$ and (W, S) is of type C_n with n odd and $n \geqslant 3$.*

If φ is an automorphism then $(M, \varphi(M))$ is a replacement pair for (W, S), and the elements of M are not intrinsic reflections of W.

Proof. The first assertion follows from [**10**, Theorem 31], in which the automorphism groups of irreducible finite Coxeter groups are described. It is obvious that $(M, \varphi(M))$ is a replacement pair if α is an automorphism, and since $|S| \geqslant 3$ it follows from assertion (iii) in Lemma 3.10 that the elements of $\varphi(M) = zM$ are not reflections. Thus, by Lemma 5.2, the elements of M are not intrinsic reflections of W. □

Proposition 6.5. *Let (W, S) be an irreducible spherical Coxeter system. If (W, S) is of type C_2, H_3, E_6, E_7, A_n for $n \neq 5$, D_n for n odd, or $I_2(n)$ for n odd or divisible by 4, then all reflections in W are intrinsic. In all other cases W has no intrinsic reflections.*

Proof. Let (W, S) be a Coxeter system of type C_2, H_3, E_6, E_7, A_n, D_n for n odd, or $I_2(n)$ for n odd or divisible by 4, and let R be a Coxeter generating set of W. By Lemma 6.3 it follows that (W, R) is irreducible and by Proposition 6.1 it then follows that there exists an automorphism α of W such that $\alpha(S) = R$. Using Theorem 31 in [**10**] and going through the cases yields $R = \alpha(S) \subseteq S^W$ except for the case where (W, S) is of type A_5. It is well known that in this particular case W is isomorphic to Sym(6) and that its exceptional automorphism is not reflection preserving. The second assertion of the Proposition follows now from Lemmas 6.4 and 6.2. □

Corollary 6.6. *If (W, S) is an irreducible spherical Coxeter system that is not of (-1)-type then all reflections in W are intrinsic, unless (W, S) is of type A_5.*

7. *Spherical principal components and blowing down*

In this section we consider those odd components M of Coxeter systems (W, S) for which the M-principal component of the even closure is spherical. Using the results of Section 6 and replacement pairs, we give conditions that ensure that the elements of M are not intrinsic reflections. A special case that arises here concerns blowing up generators. We discuss this case in some detail, because it motivates the concept of a blowing down pair for a Coxeter system. Blowing down pairs appear in assertion (i) of Theorem 1 and also play a role in the proof of Theorem 3.

We start with a general observation that will also be used in the next section.

PROPOSITION 7.1. *Let (W, S) be a Coxeter system, let M be an odd component, and let $J := C_0(M)$. Suppose that (M, M') is replacement pair for $(\langle J\rangle, J)$. Then (M, M') is a replacement pair for (W, S). Moreover, if no element of M' is conjugate to any element of M then the elements of M are not intrinsic reflections of W.*

Proof. Recall that $S \setminus M^\infty = C_0(M) \cup C_0(M)^\perp = J \cup J^\perp$ (by the remark immediately preceding Proposition 5.3). Hence the first assertion follows from Lemma 4.4.

The second assertion follows immediately from Lemma 5.2. □

COROLLARY 7.2. *Let M be an odd component of the Coxeter system (W, S) and let $J := C_0(M)$, the principal component of the even closure $E(M)$. Assume that J is spherical. Then the elements of M are not intrinsic reflections if one of the following holds.*

(i) *J is of (-1)-type and of rank at least 3, and not of type H_3 or E_7;*
(ii) *J is of type $I_2(2n)$ for some odd $n \geqslant 3$;*
(iii) *J is of type A_5.*

Proof. Suppose first that J is of (-1)-type, has rank at least 3, and is not of type H_3 or E_7. If J is of type C_n, with n odd, assume additionally that $|M| \neq 1$. Put $z = \rho_J$. It follows from Lemma 6.4 that (M, zM) is a replacement pair for $(\langle J\rangle, J)$ and the elements of zM are not reflections. Thus, by Proposition 7.1, the elements of M are not intrinsic reflections.

To complete the proof of (i) and to prove (ii), let $n \geqslant 3$ be odd, and suppose that J is of type C_n and $|M| = 1$, or that J is of type $I_2(2n)$. Let $s, a \in J$ be such that $M = \{s\}$ and $\mathrm{o}(sa) \neq 2$, and let $z := \rho_J$. Then it follows by Lemma 6.2 that $(\{s\}, \{z, sas\})$ is a replacement pair for $(\langle J\rangle, J)$. It follows from assertion (i) of Lemma 3.10 that z is not in S^w, and s is not conjugate to sas since s and a are in different odd components of S. Again, Proposition 7.1 yields that the elements of M are not intrinsic reflections.

If (iii) holds then $C_0(M)$ is of type A_5, and this forces $M = C_0(M)$ since in type A_5 systems there is only one odd component. Moreover, as $(\langle M\rangle, M)$ is of type A_5, there exists an automorphism β of $\langle M\rangle$ such that the elements of $\beta(M)$ are not conjugate to the elements of M. Applying Proposition 7.1 with $(M, M') = (M, \beta(M))$ as the replacement pair for $(\langle J\rangle, J)$, we again see that the elements of M are not intrinsic reflections. □

The arguments used in the above proof establish the following fact.

FACT. *Let (W, S) be a Coxeter system, and let M be an odd component of S consisting of a single element s. Suppose that $J := C_0(M)$ is of type $I_2(2n)$ or C_n, with n odd and $n \geqslant 3$. Let $z := \rho_J$ and let $a \in C_0(M)$ be the unique element such that $\mathrm{o}(sa) \neq 2$. Then the set $R := (S \setminus \{s\}) \cup \{z, sas\}$ is a Coxeter generating set of W, of type $A_1 + I_2(n)$ or $A_1 + D_n$.*

Thus, the existence of a generator with the properties just described implies that there is Coxeter generating set of size $|S|+1$. This is a very special situation. Generators with these properties appear in [**13**] and [**20**] in the context of the isomorphism problem for Coxeter groups. In these references they were called *pseudo-transpositions*. This way of increasing the rank by 1 was reconsidered in [**18**] under the name *blowing up*. Here we adopt this latter terminology and call the elements $s \in S$ satisfying the conditions above *blowing up generators for* (W,S).

Definition 7.3. Let (W,S) be a Coxeter system. An element $s \in S$ is called a *blowing up generator* for (W,S) if $M=\{s\}$ is an odd component of (W,S) such that $C_0(M)$ is spherical of type C_n or $I_2(2n)$, where $n \geqslant 3$ is an odd integer.

The question of reversing the blowing up construction for Coxeter generating sets was addressed by Mihalik and Ratcliffe in [**17**]. They introduced a blowing down procedure, which decreases the rank. It is slightly more general than just reversing the blowing up procedure, because it might involve twists (in the sense of [**3**]). In [**21**], in the context of intrinsic reflections, the blowing down procedure for Coxeter generating sets was reconsidered from a different angle. In that paper, a *blowing down generator for a right-angled generator* of a Coxeter system was defined. However, it seems to be more appropriate to regard the two generators as a pair, in which neither plays a privileged role.

It is convenient to make the following definition, since it makes the precise definition of blowing down pairs easier to state.

Definition 7.4. Let (W,S) be a Coxeter system. If $J \subseteq S$ then the simplicial graph Π_J is defined by

$$\Pi_J := (J, \{\{x,y\} \subseteq J \mid 2 \leqslant \mathrm{o}(xy) \neq \infty\}).$$

The graph Π_S is called the *presentation diagram of* (W,S).

Recall from Section 2 above that a right-angled generator of (W,S) is an $s \in S$ such that $\mathrm{o}(st) \in \{1,2,\infty\}$ for all $t \in S$.

Definition 7.5. Let (W,S) be a Coxeter system. A pair $(z,a) \in S \times S$ is called a *blowing down pair* for (W,S) if the following conditions hold:

(i) z is right-angled in (W,S) and $a \in z^{\perp}$;
(ii) the component C of $z^{\perp}$ containing a is of type $I_2(n)$ or D_n for some $n \geqslant 3$ with n odd, and $b := \rho_C a \rho_C \neq a$;
(iii) for each connected component U of Π_{z^∞}, either $U \subseteq a^{\infty}$ or $U \subseteq b^{\infty}$.

Definition 7.5 is reasonably compatible with the terminology of [**21**].

In (ii) of Definition 7.5 we identify $I_2(3)$ with A_2, and D_3 with A_3 renumbered (with vertex 1 in the middle). Note that C is not of (-1)-type, and so ρ_C induces the nontrivial symmetry of order 2 on its Coxeter diagram. Thus, in the $I_2(n)$ case, a is one of the two elements of C and b is the other, while in the D_n case a and b are the elements of C corresponding to vertices $n-1$ and n (the tips of the fork).

The symmetry between a and b in the above situation ensures that if (z,a) is a blowing down pair then so is (z,b).

Definition 7.6. Let (z, a) be a blowing down pair for (W, S) and let C and b be as in Definition 7.5 (ii). The *type* of (z, a) is defined to be the type of $(\langle C\rangle, C)$, and (z, a) is called a *proper blowing down pair for* (W, S) if $z^\infty \subseteq a^\infty$ or $z^\infty \subseteq b^\infty$.

The point of these definitions is that if (z, a) is a blowing down pair for (W, R) then it is possible to apply a diagram twist to $\Gamma(W, R)$ to produce a proper blowing down pair. This is proved in [**21**, Proposition 4.8].

Now let (z, a) be a proper blowing down pair for (W, S), let C and b be as in Definition 7.5 (ii), and assume that $z^\infty \subseteq b^\infty$ (which implies in fact that $z^\infty = b^\infty$). Then it is easy to check (and proved in [**21**]) that $(\{z, b\}, \{z\rho_C\})$ is a replacement pair for (W, S). Moreover, setting $R := (S \setminus \{z, b\}) \cup \{z\rho_C\})$, it follows that $z\rho_C$ is a blowing up generator for (W, R).

8. C_3-*neighbours*

In this section we consider C_3-neighbours of odd components in Coxeter systems. We shall see that the existence of a C_3-neighbour of an odd component M in a Coxeter system (W, S) implies that the elements of M are not intrinsic reflections of W. The notion of a C_3-neighbour of an odd component was introduced in [**11**], in order to describe the finite continuation of a finite order element of W. Finite continuations will be an important tool for us later.

The following lemma describes a remarkable automorphism of the Coxeter group of type C_3. It will play an important role in the theory.

Lemma 8.1. *Let (W, S) be a Coxeter system of type C_3, where $S = \{a, b, c\}$ and $\mathrm{o}(ab) = 2$, $\mathrm{o}(ac) = 3$ and $\mathrm{o}(bc) = 4$. Then there exists an automorphism α of W such that $\alpha(a) = ab$ and $\alpha(b) = b$, and α has order 2.*

Proof. It is well known and easily checked that abc has order 6; furthermore, $(abc)^3 = z$, the central involution (and longest element). So if we define $\hat{c} = zc$ and $\hat{a} = ba$ then $\hat{c}\hat{a} = (abc)^2$ has order 3. Since also $\hat{c}b = zcb$ has order 4 and $b\hat{a} = a$ has order 2, it follows from the Coxeter presentation that W has an endomorphism α mapping a and c to $\hat{a}$ and $\hat{c}$ respectively, and fixing b. We find that $\alpha(z) = \alpha((abc)^3) = (acz)^3 = (ac)^3z^3 = z$, and it follows readily that α^2 is the identity. □

Remark. With (W, S) as in Lemma 8.1, the set $M = \{a, c\}$ is an odd component of S, and it follows from Lemma 6.4 that W has an involutory automorphism φ such that $\varphi(a) = za$ and $\varphi(c) = zc$, while $\varphi(b) = b$. For each $x \in W$ let γ_x be the corresponding inner automorphism, given by $\gamma_x(w) = xwx^{-1}$ for all $w \in W$, and note that $\varphi\gamma_x\varphi = \gamma_{\varphi(x)} = \gamma_x$. Thus $\gamma_x\varphi$ has order 2 whenever x is an involution. It is readily checked that if $x = bcbc$ then $\gamma_x\varphi$ is the automorphism α defined in the proof above. In fact the conditions in Lemma 8.1 do not determine α uniquely, since $\gamma_{cbc}\varphi$ also has the required properties.

Definition 8.2. Let $M \subseteq S$ be an odd component of the Coxeter system (W, S), and let $E(M)$ be the even closure of M. A *C_3-neighbour of M* is an element $b \in S \setminus M$ such that $\mathrm{o}(bc) \in \{2, 4\}$ for all $c \in E(M) \setminus \{b\}$, with the case $\mathrm{o}(bc) = 4$ occurring for at least one c, and such that for each $c \in E(M)$ with $\mathrm{o}(bc) = 4$ there is an $a \in M$ satisfying the following conditions:

(N1) $\mathrm{o}(ba) = 2$ and $\mathrm{o}(ca) = 3$, and $\mathrm{o}(cd) = \infty$ for all $d \in M \setminus \{a, c\}$;

(N2) for all $e \in S \setminus (M \cup \{b\})$, either $\mathrm{o}(ce) = \infty$ or $\mathrm{o}(ae) = \mathrm{o}(ce) = \mathrm{o}(be) = 2$.

The set of all C_3-neighbours of M is denoted by $C_3(M)$.

In the next lemma we collect some basic observations about C_3-neighbours.

LEMMA 8.3. *Let M be an odd component of (W, S), and let $b \in S$ be a C_3-neighbour of M. Put $C := \{c \in M \mid bc \neq cb\}$. Then the following hold:*

(i) *$b \in C_0(M) \setminus M$, and if $t \in E(M)$ does not commute with b then $t \in C$;*
(ii) *C is nonempty and $C \neq M$, and every $c \in C$ has the property that $\mathrm{o}(bc) = 4$;*
(iii) *for every $c \in C$ there is a unique $a_c \in M$ such that $\mathrm{o}(ca_c) \neq \infty$;*
(iv) *if $c \in C$ and a_c are as in (iii), then $J_c := \{a_c, c, b\}$ is of type C_3;*
(v) *b commutes with all elements of $C_3(M)$.*

Proof. By the C_3-neighbour axioms, every $c \in E(M)$ such that $cb \neq bc$ satisfies $\mathrm{o}(bc) = 4$, and there is at least one such c. Moreover, for each such c there is an $a \in M$ such that $\mathrm{o}(ba) = 2$ and $\mathrm{o}(ca) = 3$. This means that $c \in M$, because it is connected to an element of M by an odd label. Since $c \in M$ and $\mathrm{o}(bc) = 4$, it follows that $b \in C_0(M)$. It also follows that $c \in C$, since $bc \neq cb$ and $c \in M$. Note that $a \notin C$, since $ba = ab$. Thus $C \neq \emptyset$ and $C \neq M$.

Since $M \subseteq E(M)$ by definition, M is an odd component of $E(M)$. Since the C_3-neighbour axioms give $\mathrm{o}(bx) \in \{1, 2, 4\}$ for all $x \in E(M)$ it follows that $\{b\}$ is an odd component of $E(M)$. So $b \notin M$. We have now proved assertions (i) and (ii).

Let $c \in C$. The existence and uniqueness of $a_c \in M$ such that $\mathrm{o}(ca_c) \neq \infty$ is immediate from (N1) in Definition 8.2, and the fact that J_c is of type C_3 then follows from (N1) together with $\mathrm{o}(bc) = 4$. Thus we are done with (iii) and (iv).

It follows from (i) that $C_3(M) \subseteq C_0(M) \setminus M$, and also that b commutes with all elements of $E(M) \setminus M$. As $C_0(M) \subseteq E(M)$, this proves (v). □

LEMMA 8.4. *Let M be an odd component of the Coxeter system (W, S) and let $b \in S$ be a C_3-neighbour of M. Then $\langle C_0(M) \rangle$ has an automorphism α such that $\alpha(e) = e$ for all $e \in C_0(M) \setminus M$ and $\alpha(a) = ab$ for all $a \in M$ that commute with b.*

Proof. Let $C = \{\, c \in M \mid bc \neq cb \,\}$ be as in assertion (ii) of Lemma 8.3, and for each $c \in C$ let a_c and the subset J_c of $C_0(M)$ be as in assertion (iv) of Lemma 8.3.

Let $c \in C$. By Lemma 8.1 there exists an automorphism α_c of $\langle J_c \rangle$ such that $\alpha_c(a_c) = ba_c$ and $\alpha_c(b) = b$. Put $\hat{c} := \alpha_c(c)$. Obviously $\hat{c}$ is an involution, since c is an involution and $\hat{c}$ its image under the automorphism α_c. Moreover, $\hat{c} \in \langle J_c \rangle$.

For each $a \in M \setminus C$ put $\hat{a} := ab$, noting that $\hat{a}$ is an involution since a and b commute. Note, moreover, that $\hat{a} = \alpha_c(a)$ for all $c \in C$ such that $a \in J_c$.

Finally, for each $e \in C_0(M) \setminus M$ put $\hat{e} := e$, noting that $\hat{b} = \alpha_c(b)$ for all $c \in C$. We claim that if s and t are in $C_0(M)$ and $m := \mathrm{o}(st)$ is finite, then $(\hat{s}\hat{t})^m = 1_W$. This claim is obviously true when $t = s$, since $\hat{s}$ is an involution in every case.

Accordingly, suppose that $t \neq s$ and $\mathrm{o}(ts) \neq \infty$, and consider first the case that $s \in C$ and $t \in C_0(M) \setminus (M \cup \{b\})$. It then follows from $(N2)$ in Definition 8.2 that t commutes with all elements of $J_s = \{a_s, s, b\}$. In particular, this implies that $m = 2$, and also that t commutes with $\hat{s}$, since $\hat{s} \in \langle J_s \rangle$. Since $\hat{t} = t$ we deduce that $(\hat{s}\hat{t})^m = (\hat{s}t)^2 = 1_W$, as required.

Next, suppose that $s \in C$ and $t \in M \cup \{b\}$. By $(N1)$ in Definition 8.2 and our assumption that $\mathrm{o}(st) \neq \infty$, it follows that $t \in \{a_s, s, b\} = J_s$. Hence $\hat{t} = \alpha_s(t)$, and $\mathrm{o}(\hat{s}\hat{t}) = \mathrm{o}(\alpha_s(s)\alpha_s(t)) = \mathrm{o}(st)$, as required.

We have now proved the claim when $s \in C$, and—by symmetry—we are left with the case $C \cap \{s, t\} = \emptyset$. So we assume henceforth that s, $t \in C_0(M) \setminus C$. Note that b commutes with both s and t, by assertion (i) of Lemma 8.3.

If s and t are both in $C_0(M) \setminus M$ then $\hat{s} = s$ and $\hat{t} = t$, and it is obvious that $\mathrm{o}(\hat{s}\hat{t}) = \mathrm{o}(st)$. If s and t are both in $M \setminus C$ then $\hat{s} = sb$ and $\hat{t} = tb$, and since $(sb)(tb) = st$ it is again obvious that $\mathrm{o}(\hat{s}\hat{t}) = \mathrm{o}(st)$. The only remaining possibility is that $s \in C_0(M) \setminus M$ and $t \in M \setminus C$, or $t \in C_0(M) \setminus M$ and $s \in M \setminus C$. In both cases $\hat{s}\hat{t} = stb$. Furthermore, s and t lie in different odd components of $C_0(M)$, and so $m = \mathrm{o}(st)$ must be even. So $(\hat{s}\hat{t})^m = (stb)^m = (st)^m b^m = 1_W$, as required, and the claim is established.

In view of the Coxeter presentation of $\langle C_0(M)\rangle$ it follows from the claim that $\langle C_0(M)\rangle$ has an endomorphism α such that $\alpha(s) = \hat{s}$ for all $s \in C_0(M)$. It remains to show that α has an inverse. It is clear that for all $s \in C$ the restriction of α to $\langle J_s\rangle$ agrees with the automorphism α_s of $\langle J_s\rangle$. Since α_s has order 2, by Lemma 8.1, we deduce that $\alpha^2(s) = s$ for all $s \in C$. For all $s \in M \setminus C$ we have $\alpha^2(s) = \alpha(sb) = \alpha(s)\alpha(b) = sb^2 = s$, and for all $s \in C_0(M) \setminus M$ we have $\alpha^2(s) = \alpha(s) = s$. So α has order 2. In particular, α is an automorphism, as required. □

PROPOSITION 8.5. *Let M be an odd component of (W, S), and suppose that M has at least one C_3-neighbour. Then the elements of M are not intrinsic reflections.*

Proof. Let b be a C_3-neighbour of M. By assertion (ii) of Lemma 8.3, there is an $a \in M$ such that $ba = ab$. Thus, by Lemma 8.4, there is an automorphism α of $\langle C_0(M)\rangle$ such that $\alpha(t) = t$ for all $t \in C_0(M) \setminus M$ and $\alpha(a) = ab$.

It follows that $(M, \alpha(M))$ is a replacement pair for $J := C_0(M)$. Moreover, as $ab = \rho_{\{a,b\}}$ it follows from assertion (i) of Lemma 3.10 that $\alpha(a) \notin S^W$, and hence $\alpha(M) \cap S^W = \emptyset$ (since all elements of $\alpha(M)$ are conjugate). Thus it follows from Proposition 7.1 that the elements of M are not intrinsic reflections. □

9. *On the proof of Theorem 1 and the finite continuation*

In the previous sections we have considered several situations in which a generator $s \in S$ of a Coxeter system (W, S) is not an intrinsic reflection of W. We summarize them in the following proposition.

PROPOSITION 9.1. *Let (W, S) be a Coxeter system, let $s \in S$ and let M be the odd component containing s. Then s is not an intrinsic reflection of W if any of the following hold:*
(i) *$E(M)$ has a subsidiary component of (-1)-type;*
(ii) *$C_0(M)$ is of (-1)-type excluding types C_2, H_3, E_7, and $I_2(4k)$ for $k \geqslant 2$;*
(iii) *$C_0(M)$ is of type A_5;*
(iv) *M has at least one C_3-neighbour.*

Proof. This follows from Proposition 5.3 for (i), from Corollary 7.2 for (ii) and (iii), and, finally, from Proposition 8.5 for (iv). □

The previous proposition finishes the proof of one direction in assertions (ii)–(iv) of Theorem 1. It is worthwhile pointing out that the rank of (W, S) is not required to be finite in this proposition. In the following sections we will provide the proof of the other direction, and here it is the case that our arguments apply only to finitely generated Coxeter groups. This is because we shall use finite continuations of finite order elements in Coxeter groups. This concept was introduced in [**11**] for finitely generated Coxeter groups. The description of the finite continuation of a generator $s \in S$ of a Coxeter system (W, S) is the key tool for the proof of Theorem 1.

In the remainder of this section we shall give the precise definition of the finite continuation and explain how it is going to be applied in this paper. We also state and prove a proposition that is a corollary of the main result in [**11**]. In this proposition we collect all the material that is needed from that paper.

DEFINITION 9.2. Suppose that (W, S) is a Coxeter system, with S finite, and let $w \in W$ be an element of finite order. The *finite continuation* of w, written $\mathrm{FC}(w)$, is defined to be the intersection of all maximal finite subgroups of W containing w.

LEMMA 9.3. *Let (W, S) be a Coxeter system of finite rank and let $w \in W$ be an element of finite order. Then the following hold.*

(i) *The finite continuation $\mathrm{FC}(w)$ is defined and is a finite parabolic subgroup of W. In particular, $\mathrm{FC}(w)$ is a finite Coxeter group;*
(ii) *If w is an intrinsic reflection of $\mathrm{FC}(w)$ then it is also an intrinsic reflection of W.*

Proof. By Proposition 3.5, each finite subgroup is contained in a finite parabolic subgroup, and so the maximal finite parabolic subgroups are precisely the maximal finite subgroups of W. Since S is finite each finite parabolic subgroup is contained in a maximal finite parabolic subgroup, and it follows from Proposition 3.1 that $\mathrm{FC}(w)$ exists and is a finite parabolic subgroup of W whenever $w \in W$ is an element of finite order.

The definition of $\mathrm{FC}(w)$ does not depend on the Coxeter generating set S. Thus, by (i), the group $\mathrm{FC}(w)$ is a parabolic subgroup with respect to any Coxeter generating set. Suppose that w is an intrinsic reflection of the Coxeter group $\mathrm{FC}(w)$ and let R be a Coxeter generating set of W. As $\mathrm{FC}(w)$ is a parabolic subgroup of (W, R) there exists a $v \in W$ and a $J \subseteq R$ such that J^v is a Coxeter generating set for $\mathrm{FC}(w)$. As w is an intrinsic reflection of $\mathrm{FC}(w)$ the element w is conjugate in $\mathrm{FC}(w)$ to an element of J^v, which is a subset of R^W. Hence $w \in R^W$. □

We are now able to explain in more detail how the finite continuation is used in the proof of Theorem 1. By Proposition 9.1 the task is to show that $s \in S$ is an intrinsic reflection of W under certain conditions. By Lemma 9.3 (ii) it suffices to show that s is an intrinsic reflection in its finite continuation. For this, we need enough information about $\mathrm{FC}(s)$, and criteria ensuring that a generator of a finite, possibly reducible, Coxeter group is an intrinsic reflection. The latter will be presented in the next section. In the remainder of this section we describe the information on $\mathrm{FC}(s)$ that we need, explaining how to obtain it from [**11**].

PROPOSITION 9.4. *Let (W, S) be a Coxeter system of finite rank and let M be an odd component of (W, S). Then the following hold.*

(i) *If L is a spherical component of $E(M)$ then $\langle L \rangle \subseteq \mathrm{FC}(s)$ for each $s \in M$.*
(ii) *There exists an $s \in M$ such that $\mathrm{FC}(s) = \langle J \rangle$ for some spherical subset J of S such that $s \in J$.*

Proof. Assertion (i) is Lemma 36 in [**11**]. The existence of a subset J of S such that $\mathrm{FC}(s) = \langle J \rangle$ is the very first assertion of Theorem 7 in [**11**] and the argument is given in the first paragraph of Section 4 of that paper. That J is spherical is clear from the finiteness of $\mathrm{FC}(s)$. Finally, as $s \in \mathrm{FC}(s) = \langle J \rangle$ we have $s \in \langle J \rangle \cap S = J$. □

Corollary 9.5. *Let (W,S) be a Coxeter system of finite rank, let M be an odd component of (W,S), and let $s \in M$ and $J \subseteq S$ be such that $\mathrm{FC}(s) = \langle J \rangle$. Then each spherical component of $E(M)$ is a component of J.*

Proof. It follows from assertion (ii) of Proposition 9.4 that J is spherical and $s \in J$. Thus $J \subseteq S \setminus s^\infty \subseteq S \setminus M^\infty = E(M)$. Let L be a spherical component of $E(M)$. By assertion (i) of Proposition 9.4, we have $\langle L \rangle \subseteq \mathrm{FC}(s) = \langle J \rangle$. This yields $L \subseteq J$, since $\langle L \rangle \cap \langle J \rangle = \langle L \cap J \rangle$. Clearly $J \setminus L \subseteq L^\perp$, since $E(M) \setminus L \subseteq L^\perp$, and as L is irreducible we conclude that L is a component of J. □

Notation. If M is an odd component of the Coxeter system (W,S), we let $\Sigma(M)$ denote the union of all spherical M-subsidiary components of $E(M)$.

Note that $C_0(M)$ and $\Sigma(M)$ are disjoint direct factors of $E(M)$, and the non-spherical M-subsidiary components are also direct factors of $E(M)$.

Let (W,S) be a Coxeter system of finite rank and M an odd component of (W,S). Let $s \in M$ and $J \subseteq S$ be such that $\mathrm{FC}(s) = \langle J \rangle$. By Corollary 9.5 we know that $\Sigma(M)$ is a union of components of J. Thus, in order to have a precise description of $\mathrm{FC}(s)$ it remains to describe $J \setminus \Sigma(M)$. Our next proposition does this.

Recall that the (usually empty) set of all C_3-neighbours of an odd component M is denoted by $C_3(M)$.

Proposition 9.6. *Let (W,S) be a Coxeter system of finite rank, and let M be an odd component of (W,S). Choose an element $s \in M$ such that $\mathrm{FC}(s) = \langle J \rangle$ for some $J \subseteq S$, and put $K := J \setminus \Sigma(M)$. Then $s \in K \subseteq C_0(M)$, and the following hold.*

(i) *The set K is the whole of $C_0(M)$ if and only if $C_0(M)$ is spherical.*

(ii) *If $C_0(M)$ is not spherical then one of the following holds:*
  (a) *$K = \{s\} \cup C_3(M)$ and is of type $(k+1)A_1$, where $k = |C_3(M)|$;*
  (b) *K is of type C_2;*
  (c) *K is of type $2A_1$ and $K \subseteq M$.*

Proof. This follows from Theorem 7 in [**11**]. In that theorem four cases A, B, C and D are listed. Assertion (i) corresponds to Case A, while parts (a), (b) and (c) of assertion (ii) correspond to B, C and D respectively.

Case D of [**11**, Theorem 7] is distinguished from the other cases by the fact that $C_0(M)$ is non-spherical and K contains two elements of M. The fact that they commute is one of the assertions of the theorem, in view of [**11**, Definition 6].

Case C of [**11**, Theorem 7] is distinguished from the others by the fact that $C_0(M)$ is non-spherical and $K = \{s,t\}$ for some t such that $st \neq ts$. The fact that K is of type C_2 and $t \in C_0(M)$ is an assertion of the theorem, in view of [**11**, Definition 5].

Case B of [**11**, Theorem 7] is distinguished from the others by the fact that $C_0(M)$ is non-spherical and the elements of $K \setminus M$ (if any) are C_3-neighbours of M. Since $\mathrm{FC}(s)$ is visible, it is one of the assertions of the theorem that s is not adjacent in $\Gamma(\langle C_0(M) \rangle, C_0(M))$ to any C_3-neighbour of M. Hence s commutes with all C_3-neighbours of M, by (i) of Lemma 8.3, and the C_3-neighbours all commute with each other, by (v) of Lemma 8.3. Also $C_3(M) \subseteq C_0(M)$, by (i) of Lemma 8.3. This justifies all of our claims in (ii) (a). □

We shall need the following corollary in the proof of Proposition 11.2.

Corollary 9.7. *Let (W, S) be a Coxeter system of finite rank and let M be an odd component of (W, S). Choose $s \in M$ such that $\mathrm{FC}(s) = \langle J \rangle$ for some $J \subseteq S$. If the centre of $\mathrm{FC}(s)$ is the subgroup generated by s then each component of $(\langle J \rangle, J)$ of rank at least two is an M-subsidiary component of $E(M)$.*

Proof. Since $|Z(\langle J \rangle)| = 2$ it follows that J has exactly one component of (-1)-type, and s is the longest element of the corresponding visible subgroup (see Section 3 above). Thus $\{s\}$ is a component of J, and the only component of type A_1. By Proposition 9.6 it follows that if $C_0(M)$ is spherical then $C_0(M) = \{s\}$, while if $C_0(M)$ is non-spherical then case (ii) a) holds with $k = 0$. In either case every component of J of rank at least two is contained in $\Sigma(M)$, and hence is an M-subsidiary component of $E(M)$. □

10. *Intrinsic reflections in reducible spherical Coxeter systems*

In order to analyse reducible spherical Coxeter systems (W, S) it is useful to recall some basic results on direct product decompositions of groups, and, in particular, the Krull–Remak–Schmidt Theorem.

Definition 10.1. A group G is called *indecomposable* if it has order greater than 1 and is not the inner direct product of two proper subgroups.

A family $(H_i)_{1 \leqslant i \leqslant n}$ of subgroups of G is called a *Remak decomposition* of G if each H_i is indecomposable and G is the inner direct product $H_1 \times H_2 \times \cdots \times H_n$.

The following is obvious.

Lemma 10.2. *Any finite group admits a Remak decomposition.*

Definition 10.3. An automorphism α of a group is called a *central automorphism of G* if α induces the identity on $G/Z(G)$, that is, if $\alpha(g) \in gZ(G)$ for all $g \in G$.

We shall need the following version of the Krull–Remak–Schmidt Theorem.

Proposition 10.4. *Let $(H_i)_{1 \leqslant i \leqslant n}$ and $(K_j)_{1 \leqslant j \leqslant m}$ be Remak decompositions of the finite group G. Then $n = m$ and there exists a permutation $\pi \in \mathrm{Sym}(n)$ and a central automorphism α of G such that $\alpha(H_i) = K_{\pi(i)}$ for all $i \in \{1, 2, \ldots, n\}$.*

Proof. This is a special case of Theorem 3.3.8 in [**25**]. □

Corollary 10.5. *Suppose that $(H_i)_{1 \leqslant i \leqslant m}$ and $(K_i)_{1 \leqslant i \leqslant n}$ are Remak decompositions of the finite group G, and suppose also that $Z(G) \subseteq H_1$. Then $H_1 = K_j$ for some $j \in \{1, 2, \ldots, n\}$. In particular, if the centre of G is trivial, then there exists a permutation $\pi \in \mathrm{Sym}(n)$ such that $K_i = H_{\pi(i)}$ for all $i \in \{1, 2, \ldots, n\}$.*

Proof. Since $Z(G)$ is contained in H_1, it follows that each central automorphism of G stabilizes H_1. So the assertion follows from Proposition 10.4. □

LEMMA 10.6. *Let (W, S) be a Coxeter system and J a union of components of (W, S). If $s \in J$ is an intrinsic reflection of W then it is also an intrinsic reflection of $\langle J \rangle$.*

Proof. Since J is a union of components of (W, S) it follows that $W = X \times Y$, where $X = \langle S \setminus J \rangle$ and $Y = \langle J \rangle$. Suppose that $s \in J$ is an intrinsic reflection of W, and let K be any Coxeter generating set for Y. Then $R := (S \setminus J) \cup K$ is a Coxeter generating set for W, by Proposition 4.1, and $s \in R^W$ since s is an intrinsic reflection of W. But $R^W = (S \setminus J)^X \cup K^Y$, and $(S \setminus J)^X \subseteq X$. Since s is a non-identity element of Y it follows that $s \in K^Y$, as required. □

PROPOSITION 10.7. *Let (W, S) be a spherical Coxeter system, and let $s \in S$. If $|Z(W)| \geqslant 4$ then s is not an intrinsic reflection, and if $|Z(W)| = 1$ then s is an intrinsic reflection if and only if the $\{s\}$-principal component of S is not of type A_5.*

Proof. Let M be the odd component of S containing s. Since S is spherical it is clear that $E(M) = S$. Observe that $C_0(M)$ is the $\{s\}$-principal component of S, and the other components of S are the M-subsidiary components.

If $|Z(W)| \geqslant 4$ then S has at least two components of (-1)-type, and hence at least one M-subsidiary component of (-1)-type. So s is not an intrinsic reflection of W, by Proposition 5.3.

Now suppose that $|Z(W)| = 1$, and let $J_1, J_2, \ldots, J_n$ be the components of S, numbered so that $J_1 = C_0(M)$. If s is an intrinsic reflection of W then it is an intrinsic reflection of $\langle J_1 \rangle$, by Lemma 10.6. Thus J_1 is not of type A_5, by Proposition 6.5.

Conversely, suppose that J_1 is not of type A_5, and let R be an arbitrary Coxeter generating set for W. Let $K_1, K_2, \ldots, K_m$ be the components of R. Since $|Z(W)| = 1$, no J_i or K_j is of (-1)-type, and so the $\langle J_i \rangle$ and $\langle K_j \rangle$ are all indecomposable, by Lemma 6.3. Thus $(\langle J_i \rangle)_{1 \leqslant i \leqslant n}$ and $(\langle K_j \rangle)_{1 \leqslant j \leqslant m}$ are both Remak decompositions of W. Since $Z(W)$ is trivial there are no nontrivial central automorphisms of W, and so $\langle J_1 \rangle = \langle K_j \rangle$ for some j, by Proposition 10.4. Thus J_1 and K_j have the same type, by Proposition 6.1. Since this type is not A_5 it follows from Corollary 6.6 that all elements of J_1 are intrinsic reflections of $\langle J_1 \rangle$. In particular, s is conjugate to an element of K_j. So $s \in R^W$, and hence s is an intrinsic reflection of W, as required. □

In view of Proposition 10.7 and the Fact stated after the third paragraph of Section 3, the remaining task for spherical Coxeter systems is to clarify the situation when $|Z(W)| = 2$. In this case, every Coxeter generating set for W includes exactly one component of (-1)-type.

PROPOSITION 10.8. *Let (W, S) be a Coxeter system with $|Z(W)| = 2$, and let $s \in S$. If the $\{s\}$-principal component of S is not of (-1)-type then s is not an intrinsic reflection of W.*

Proof. This is immediate from Proposition 5.3, since the component of S of (-1)-type is an M-subsidiary component of $E(M) = S$ (where M is the odd component containing s). □

For finite Coxeter groups W with $|Z(W)| = 2$, it remains to determine whether or not the elements of the unique component of (-1)-type are intrinsic reflections.

PROPOSITION 10.9. *Let (W, S) be a spherical Coxeter system with $|Z(W)| = 2$, and let $s \in S$. If the $\{s\}$-principal component of S is of type H_4, F_4, E_8, $I_2(2n)$ for n odd, C_n for $n > 2$, or D_n for n even and $n \geqslant 4$, then s is not an intrinsic reflection of W.*

Proof. This is an immediate consequence of Propositions 10.6 and 6.5. □

PROPOSITION 10.10. *Let (W, S) be a spherical Coxeter system with $|Z(W)| = 2$, and let $s \in S$. Let $\widehat{J}$ be the $\{s\}$-principal component of S. Then s is an intrinsic reflection of W if $\widehat{J}$ is of type C_2, H_3, E_7 or $I_2(4k)$, while if $\widehat{J}$ is of type A_1 then s is an intrinsic reflection of W if and only if S has no component of type $I_2(n)$ or D_n with n odd and $n \geqslant 3$.*

Proof. Suppose that $\widehat{J}$ is of type A_1, so that $\widehat{J} = \{s\}$, and suppose that S also has a component of type $I_2(n)$ or D_n, where n is odd. Let J be the union of this component with $\{s\}$, so that J is of type $A_1 + I_2(n)$ or $A_1 + D_n$. As we have seen in Lemmas 6.2 and 6.3, it follows that the group $\langle J \rangle$ is isomorphic to an irreducible Coxeter group of type $I_2(2n)$ or C_n, and has no intrinsic reflections (by Proposition 6.5). So s is not an intrinsic reflection of W, by Proposition 10.6.

It remains to show that if $\widehat{J}$ is of type C_2, H_3, E_7, $I_2(4k)$ or A_1 then s is an intrinsic reflection, provided in the A_1 case that S does not also have a component of type $I_2(n)$ or D_n with n odd. Accordingly, we assume that these conditions are satisfied, aiming to show that $s \in R^W$ for all subsets $R \subseteq W$ such that (W, R) is a Coxeter system.

Suppose that $R \subseteq W$ and (W, R) is a Coxeter system. Let $\widehat{J} = J_1, J_2, \ldots, J_h$ be the components of S, and let $K_1, K_2, \ldots, K_m$ be the components of R. Since $|Z(W)| = 2$, exactly one of the K_j is of (-1)-type, and we can choose the numbering so that it is K_1. Write $\mathcal{G}_i = \langle J_i \rangle$ for $2 \leqslant i \leqslant h$ and $\mathcal{H}_j = \langle K_j \rangle$ for $2 \leqslant j \leqslant m$, and note that all of these groups are indecomposable (by Lemma 6.3). If the group $\langle J_1 \rangle$ is decomposable then by Lemma 6.3 we can write $\langle J_1 \rangle = \mathcal{G}_0 \times \mathcal{G}_1$ where $\mathcal{G}_0 = Z(\langle J_1 \rangle)$ and $\mathcal{G}_1$ is indecomposable; if $\langle J_1 \rangle$ is indecomposable we put $\mathcal{G}_1 = \langle J_1 \rangle$. Similarly, if the group $\langle K_1 \rangle$ is decomposable we write $\langle K_1 \rangle = \mathcal{H}_0 \times \mathcal{H}_1$ with $\mathcal{H}_0 = Z(\langle K_1 \rangle)$ and $\mathcal{H}_1$ indecomposable, and if $\langle K_1 \rangle$ is indecomposable we put $\mathcal{H}_1 = \langle K_1 \rangle$. We shall show that $\langle J_1 \rangle = \langle K_1 \rangle$ in every case.

If J_1 is of type C_2 or $I_2(4k)$ then $\langle J_1 \rangle$ is indecomposable (by Lemma 6.3), and $(\mathcal{G}_i)_{1 \leqslant i \leqslant h}$ is a Remak decomposition for W with $|\mathcal{G}_i| > 2$ for all i. It follows that $\langle K_1 \rangle$ is indecomposable, or else $(\mathcal{H}_j)_{0 \leqslant j \leqslant m}$ would be a Remak decomposition of W with $|\mathcal{H}_0| = 2$, contradicting Proposition 10.4. So $\langle J_1 \rangle = \langle K_1 \rangle$, by Corollary 10.5.

If J_1 is of type H_3 or E_7 then Lemma 6.3 tells us that $\langle J_1 \rangle$ decomposes as $\mathcal{G}_0 \times \mathcal{G}_1$, where $\mathcal{G}_0$ has order 2 and $\mathcal{G}_1$ is not a Coxeter group, and hence $(\mathcal{G}_i)_{0 \leqslant i \leqslant h}$ is a Remak decomposition of W with $\mathcal{G}_0$ of order 2 and $\mathcal{G}_1$ not a Coxeter group. So $\langle K_1 \rangle$ must also decompose, or else $(\mathcal{H}_j)_{1 \leqslant j \leqslant m}$ would be a Remak decomposition of W with all factors Coxeter groups. Moreover, the decomposition of $\langle K_1 \rangle$ must yield a factor that is not a Coxeter group. So $\langle K_1 \rangle = \mathcal{H}_0 \times \mathcal{H}_1$ with $\mathcal{H}_0$ of order 2 and $\mathcal{H}_1$ not a Coxeter group, and by Proposition 10.4 there is a central automorphism α of W such that $\alpha(\mathcal{G}_1) = \mathcal{H}_1$ (since the other factors are Coxeter groups). Since $\mathcal{G}_0 = Z(W) = \mathcal{H}_0$ and the automorphism α is central, we conclude that $\langle J_1 \rangle = \mathcal{G}_0\mathcal{G}_1 = \mathcal{G}_0\alpha(\mathcal{G}_1) = \mathcal{H}_0\mathcal{H}_1 = \langle K_1 \rangle$.

Finally, if J_1 is of type A_1 then $(\mathcal{G}_i)_{1 \leqslant i \leqslant h}$ is a Remak decomposition of W such that $\mathcal{G}_1$ has order 2 and all the $\mathcal{G}_i$ are Coxeter groups. Moreover, by our extra information in this case, and using Proposition 6.1, no $\mathcal{G}_i$ is isomorphic to a Coxeter group of type $I_2(n)$ or D_n with n odd. It follows that $\langle K_1 \rangle$ is indecomposable, since, by Lemma 6.3, a decomposition of $\langle K_1 \rangle$ would produce a Remak decomposition $(\mathcal{H}_j)_{0 \leqslant j \leqslant m}$ with $\mathcal{H}_1$ either not a Coxeter group (if K_1 is of type H_3 or E_7) or a Coxeter group of type $I_2(n)$ or D_n with n odd (if K_1 is of type $I_2(2n)$ or C_n with n odd). So in this case we must have $\langle J_1 \rangle = Z(W) = \langle K_1 \rangle$.

Since $\langle J_1 \rangle = \langle K_1 \rangle$, and s is an intrinsic reflection of $\langle J_1 \rangle$ by Proposition 6.5, it follows that s is conjugate to an element of K_1, and so $s \in R^W$, as desired. □

11. *Finite continuations and intrinsic reflections*

We commence this section with a technical lemma that is needed for the proof of the subsequent proposition. Note that if W is a Coxeter group then obviously any conjugate of a Coxeter generating set for W is also a Coxeter generating set for W.

LEMMA 11.1. *Let (W, S) be a Coxeter system with S finite and let $s \in S$. Let $R \subseteq W$ be a Coxeter generating set for W. Then there exists a Coxeter generating set R_1 and $L \subseteq K \subseteq R_1$ such that the following hold:*
(i) $R_1 = R^w$ *for some* $w \in W$;
(ii) $\mathrm{FC}(s) = \langle K\rangle$;
(iii) $(\langle L\rangle, L)$ *is of* (-1)*-type and* $s = \rho_L$.

Proof. Since the subgroup $\mathrm{FC}(s)$ must be parabolic relative any Coxeter generating set in W, it follows that $\mathrm{FC}(s) = \langle M\rangle^x$ for some $M \subseteq R$ and some $x \in W$. Let $K' := M^x$ and $R' := R^x$, noting that (W, R') is a Coxeter system. Since $s \in \langle K'\rangle$ is an involution it follows from Proposition 3.8 that there exist $y \in \langle K'\rangle$ and $L' \subseteq K'$ such that L' is of (-1)-type and $s = \rho^y$, where ρ is the element of $\langle L'\rangle$ whose length relative to L' is maximal. Defining $L := (L')^y$ gives $s = \rho_L$. Furthermore, $L \subseteq K \subseteq R_1$, where $K := (K')^y$ and $R_1 := (R')^y = R^{xy}$, and $\mathrm{FC}(s) = \langle K'\rangle = \langle K'\rangle^y = \langle K\rangle$. □

PROPOSITION 11.2. *Let (W, S) be a Coxeter system with S finite, and let $s \in S$. Suppose that s generates $Z(\mathrm{FC}(s))$. Then one of the following holds:*
(i) *s is an intrinsic reflection of W,*
(ii) *s is a right-angled element of S.*

Proof. Since $\mathrm{FC}(s)$ is a parabolic subgroup of W, there exist a subset $J \subseteq S$ and an element $w \in W$ such that $\mathrm{FC}(s) = w\langle J\rangle w^{-1}$. Observe that this gives $\mathrm{FC}(s^w) = \langle J\rangle$. By Lemma 3.3, since $s^w \in \mathrm{FC}(s^w)$, there exist $v \in \langle J\rangle$ and $r \in J$ such that $r = s^{wv}$. Since $Z(\mathrm{FC}(s)) = \langle s\rangle$ by hypothesis, it follows that $Z(\mathrm{FC}(r)) = Z(\mathrm{FC}(s))^{wv} = \langle s\rangle^{wv} = \langle r\rangle$. Moreover, $r \in J$, and $\mathrm{FC}(r) = \mathrm{FC}(s^w)^v = \langle J\rangle^v = \langle J\rangle$. Note that r and s belong to the same odd component M of S, since they are conjugate in W.

Since $r \in J$ and $r \in Z(\langle J\rangle)$ the set $\{r\}$ is a component of J. Let $J' = J \setminus \{r\}$, the union of the other components. Then $\mathrm{FC}(r) = \langle r\rangle \times \langle J'\rangle$. Note that $|Z(\langle J'\rangle)| = 1$, since $|Z(\mathrm{FC}(r))| = 2$.

Suppose that s is not an intrinsic reflection of W. Then r is also not an intrinsic reflection of W, since it is conjugate to s. Hence W has a Coxeter generating set R such that $r \notin R^W$. By Lemma 11.1, replacing R by a conjugate of itself, we may assume that $r = \rho_L$ and $\mathrm{FC}(r) = \langle K\rangle$, where $L \subseteq K \subseteq R$ and L is of (-1)-type.

We now show that every $x \in K \setminus L$ commutes with every $y \in L$. Let $x \in K \setminus L$ be arbitrary. Since $\ell(x) = 1$, where ℓ is length relative to R, clearly x is the minimal length element of the double coset $\langle L\rangle x\langle L\rangle$. Thus Proposition 3.1 gives $\langle L\rangle \cap x\langle L\rangle x^{-1} = \langle I\rangle$, where $I = L \cap xLx^{-1}$, which is the set of all $y \in L$ that commute with x (since $\ell(xyx) = 3$ if $\mathrm{o}(yx) \neq 2$). Since r is central in $\mathrm{FC}(r) = \langle K\rangle$ by hypothesis, $r = xrx^{-1} \in \langle L\rangle \cap x\langle L\rangle x^{-1} = \langle I\rangle$. If I were a proper subset of L this would contradict the fact that r is the longest element of $\langle L\rangle$. So $I = L$, and we have proved our claim that all $x \in K \setminus L$ commute with all $y \in L$. Thus L is a union of components of K.

Suppose, for a contradiction, that L is not irreducible, and let L_1 be a component of L. Since L is of (-1)-type, so is L_1, and the longest element of L_1 is in the centre of $\langle L\rangle$, and hence also in the centre of $\langle K\rangle$. This contradicts the fact that r is the only nontrivial element of

$Z(\mathrm{FC}(r)) = Z(\langle K\rangle)$. So L is irreducible, and hence is a component of K. Since $\rho_L = r \notin R^W$, it follows that $|L| > 1$.

Since $\langle L\rangle$ is a subgroup of $\mathrm{FC}(r)$ and $\langle J'\rangle$ is a normal subgroup of $\mathrm{FC}(r)$ of index 2, the group $X := \langle L\rangle \cap \langle J'\rangle$ is a normal subgroup of $\langle L\rangle$ of index at most 2. In fact, the index is exactly 2, since $r = \rho_L$ is in L but not in $\langle J'\rangle$. Thus $\langle L\rangle = \langle r\rangle \times X$. As $(\langle L\rangle, L)$ is an irreducible spherical Coxeter system, the group X is indecomposable, by Lemma 6.3. As X is a direct factor of $\langle L\rangle$, which is a direct factor of $\mathrm{FC}(r)$, the subgroup X is also a direct factor of $\mathrm{FC}(r)$. As it is contained in the subgroup $\langle J'\rangle$ of $\mathrm{FC}(r)$, it is a direct factor of $\langle J'\rangle$. The latter is a finite Coxeter group with trivial centre, and therefore a direct product of indecomposable Coxeter groups, by Lemma 6.3. By Proposition 10.4 (Krull–Remak–Schmidt) these Coxeter groups are the only indecomposable direct factors of $\langle J'\rangle$. So $X = \langle J_1\rangle$, where J_1 is a component of J'.

Note that $|J_1| > 1$, since $\langle J'\rangle$ has trivial centre, and that J_1 is a component of J, since J' is a union of components of J. It follows now from Corollary 9.7 that J_1 is an M-subsidiary component of $E(M)$, where M is the odd component containing r.

We now show that r is a right-angled element of S, that is, $\mathrm{o}(rt) \in \{2, \infty\}$ for all $t \in S \setminus \{r\}$. We know already that $\mathrm{o}(rt) = 2$ for all $t \in J' = J \setminus \{r\}$, since r centralizes $\langle J'\rangle$.

Let $t \in S \setminus J$ be such that $\mathrm{o}(rt) \neq \infty$. Then $t \in E(M)$, since $r \in M$, and $t \notin J_1$, since $J_1 \subseteq J$. Since J_1 is a component of $E(M)$ it follows that t centralizes $\langle J_1\rangle = X$, and since $X \subseteq \langle L\rangle$ it follows that $X \subseteq \langle L\rangle \cap t\langle L\rangle t^{-1}$, a parabolic subgroup of $\langle L\rangle$. Since X has index 2 in $\langle L\rangle$, it follows that $\langle L\rangle \cap t\langle L\rangle t^{-1}$ is also normal in $\langle L\rangle$, and therefore must be the whole of $\langle L\rangle$ since L is irreducible. Thus t normalizes $\langle L\rangle$, and hence centralizes $\rho_L = r$. So $\mathrm{o}(rt) = 2$, as required. Thus r is right-angled.

Since r is right-angled, it is clearly the only element of its odd component in S. Hence $r = s$, and it follows that s is a right-angled element of S. □

PROPOSITION 11.3. *Let (W, S) be a Coxeter system of finite rank and let $s \in S$. Suppose that $Z(\mathrm{FC}(s)) = \langle s, t\rangle$ for some $t \neq s$ such that t is conjugate to s in W. Then s is an intrinsic reflection of W.*

Proof. By our assumptions the group $Z(\mathrm{FC}(s))$ is elementary abelian of order 4, and its three non-trivial elements are s, t and $st = ts$.

Let $R \subseteq W$ be a Coxeter generating set of W. We have to show that $s \in R^W$, and by Lemma 11.1 we may assume that $\mathrm{FC}(s) = \langle K\rangle$ for some $K \subseteq R$.

Since $|Z(\mathrm{FC}(s))| = 4$, exactly two components of K are of (-1)-type. If we denote these components by P and Q, then $Z(\mathrm{FC}(s)) \setminus \{1\} = \{\rho_P, \rho_Q, \rho_{P\cup Q}\} = \{s, t, st\}$.

Since s and t are conjugate in W they must have the same rank relative to (W, R), and since the rank of $\rho_{P\cup Q}$ is clearly the sum of the ranks of ρ_P and ρ_Q, it follows that $\{s, t\} = \{\rho_P, \rho_Q\}$. Now by Lemma 3.11 we can deduce that $\langle P\rangle$ and $\langle Q\rangle$ are conjugate subgroups. Furthermore, since $t \neq s$ we have $\rho_P \neq \rho_Q$, and it follows from Proposition 3.12 that $|P| = |Q| = 1$. Hence s and t are in R^W. □

12. *Proof of Theorem 1*

Let (W, S) be a Coxeter system of finite rank, M an odd component of (W, S) and $s \in M$. In Proposition 9.1 we have established that s is not an intrinsic reflection of W if $E(M)$ has a subsidiary component of (-1)-type, which is the first claim in Theorem 1, and also the "only if" implications in (ii), (iii) and (iv). That is, s is not an intrinsic reflection of W if $C_0(M)$ is non-spherical and M has a C_3-neighbour, or if $C_0(M)$ is spherical and is of one of the following types: C_n (for $n \geqslant 3$), D_{2n} (for $n \geqslant 2$), F_4, H_4, E_8, $I_2(4n+2)$ (for $n \geqslant 1$), A_5.

Part (i) of Theorem 1 is the main result of [**21**]: if $C_0(M)$ is of type A_1 and $E(M)$ has no subsidiary components of (-1)-type then s is an intrinsic reflection of W if and only if there is no $t \in S$ such that (s,t) is a blowing down pair for (W,S).

Hence it remains to establish the "if" implications of parts (ii), (iii) and (iv): if $E(M)$ has no subsidiary components of (-1)-type then s is an intrinsic reflection of W if $C_0(M)$ is non-spherical and has no C_3-neighbours, or if $C_0(M)$ is spherical and not of one of the types mentioned above.

Accordingly, for the remainder of this section we assume that $E(M)$ has no subsidiary components of (-1)-type. Recall that $s \in M$ is an intrinsic reflection of W if and only if the same holds for each $t \in M$. Hence we are free to choose s to be whichever element of M is most convenient. By Proposition 9.4 (ii) there exist $s \in M$ and $L \subseteq S$ such that $\mathrm{FC}(s) = \langle L \rangle$. So for the remainder of this section we fix s and L so that this condition is satisfied. Recall that $s \in L$ and that L is a spherical subset of S.

Let $\Sigma(M)$ denote the union of all M-subsidiary spherical components of $E(M)$.

Proof of parts (ii) and (iii) of Theorem 1. Assume that $C_0(M)$ is spherical. Then $C_0(M)$ is a component of L, and $L = C_0(M) \cup \Sigma(M)$, by Proposition 9.6. Since $\Sigma(M)$ has no components of (-1)-type, we see that $Z(\mathrm{FC}(s)) = Z(\langle L \rangle) = Z(\langle C_0(M) \rangle)$. Thus $|Z(\mathrm{FC}(s))| = 2$ if $C_0(M)$ is of (-1)-type, and $|Z(\mathrm{FC}(s))| = 1$ otherwise.

If $C_0(M)$ is of type C_2, H_3, E_7 or $I_2(4k)$ then applying Proposition 10.10 with $(\mathrm{FC}(s), L)$ in place of (W,S) shows that s is an intrinsic reflection of $\mathrm{FC}(s)$, and thus an intrinsic reflection of W, by Lemma 9.3. This completes the proof of (ii).

If $C_0(M)$ is not of (-1)-type and not of type A_5 then applying Proposition 10.7 with $(\mathrm{FC}(s), L)$ in place of (W,S) shows that s is an intrinsic reflection of $\mathrm{FC}(s)$, and an intrinsic reflection of W, by Lemma 9.3. This completes the proof of (iii). □

Proof of part (iv) of Theorem 1. Suppose that $C_0(M)$ is not spherical and that M has no C_3-neighbours. Let $K := L \cap C_0(M)$. It follows from Proposition 9.6 that $L = K \cup \Sigma(M)$, and there are three possibilities for K: either $K = \{s\}$, or K is of type C_2, or K is of type $2A_1$ and $K \subseteq M$. Note that $Z(\mathrm{FC}(s)) = Z(\langle L \rangle) = Z(\langle K \rangle)$, since $E(M)$ has no M-subsidiary components of (-1)-type.

Suppose that $K = \{s\}$, so that $Z(\mathrm{FC}(s)) = \langle s \rangle$. Note that if s is a right-angled element of S then $C_0(M) = M = \{s\}$, contradicting the hypothesis that $C_0(M)$ is non-spherical. It follows from Proposition 11.2 that s is an intrinsic reflection of W, as desired.

If K is of type C_2 then $|Z(\mathrm{FC}(s))| = |Z(\langle K \rangle)| = 2$, and applying Proposition 10.10 with $(\mathrm{FC}(s), L)$ in place of (W,S) shows that s is an intrinsic reflection of $\mathrm{FC}(s)$. So s is an intrinsic reflection of W, as desired.

If K is of type $2A_1$ and $K \subseteq M$, then there exists $t \in M$ such that $t \neq s$ and $st = ts$, and $Z(\mathrm{FC}(s)) = \langle s,t \rangle$. Since s is conjugate to all elements of the odd component M, it follows that $t = s^w$ for some $w \in W$. Thus Proposition 11.3 applies, and we conclude that s is an intrinsic reflection of W in this case as well. □

13. *Twist rigid Coxeter systems*

13.1. *Definitions and preliminary results*

Let (W,S) be a Coxeter system. For each $J \subseteq S$ we define $J^* := S \setminus (J \cup J^\perp)$. That is, $J^* = \{\, t \in S \setminus J \mid \mathrm{o}(st) \neq 2 \text{ for some } s \in J \,\}$. Set $\mathcal{P}(J) := \{\, \{x,y\} \subseteq J \mid 2 \leqslant \mathrm{o}(xy) \neq \infty \,\}$, and recall from Definition 7.4 that $\mathcal{P}(J)$ is the edge set of Π_J, the presentation diagram of $(\langle J \rangle, J)$. A subset K of S is called an *edge of* (W,S) if $K \in \mathcal{P}(S)$.

Note that if $J \subseteq S$ is irreducible then $J^* = \{\, a \in S \setminus J \mid J \cup \{a\} \text{ is irreducible} \,\}$. In general, if $J \subseteq S$ and $a \in J^*$ then the number of components of $J \cup \{a\}$ is no greater than the number of components of J.

Definition 13.1. A Coxeter system is called *twist rigid* if the graph Π_{J^*} is connected for each irreducible spherical subset J of S.

Remark. It is clear that a Coxeter system that is not twist rigid admits a proper twist. Thus, by Proposition 4.6, such systems are not strongly rigid.

Proposition 13.2. *Let (W, S) be a twist rigid Coxeter system and let $K \subseteq S$ be irreducible and spherical. Let $s \in K$ be such that $\mathrm{o}(xs) = \infty$ for all $x \in K^*$. Then the following hold:*

(i) $S \setminus K \subseteq s^\perp \cup s^\infty$;
(ii) $S \setminus K \subseteq t^\perp$ *for every* $t \in K \setminus \{s\}$.

Proof. Since $K^* \subseteq s^\infty$ and $S \setminus K = K^\perp \cup K^*$, it follows that $S \setminus K \subseteq s^\perp \cup s^\infty$. Thus (i) holds. Since $S \setminus K = K^\perp \cup K^*$, and clearly $K^\perp \subseteq t^\perp$ for every $t \in K \setminus \{s\}$, the remaining task is to show that $K^* \subseteq t^\perp$ for every $t \in K \setminus \{s\}$.

Assume, for a contradiction, that there exists $t \in K \setminus \{s\}$ and $x \in K^*$ such that $\mathrm{o}(tx) \neq 2$. Let $K' := K \setminus \{s\}$ and let $J \subseteq K'$ be the irreducible component of $(\langle K' \rangle, K')$ containing t. Then clearly $J \cup \{s\}$ is irreducible, and therefore $s \in J^*$. As $\mathrm{o}(tx) \neq 2$ and $x \in S \setminus K \subseteq S \setminus J$, it follows that $x \in J^*$ as well. Because J is irreducible and spherical, and the system (W, S) is twist rigid, the graph Π_{J^*} is connected. Let $x = x_0, x_1, \ldots, x_k = s$ be a path of minimal length from x to s in Π_{J^*}, and put $y := x_{k-1}$. Note that $y \in J^* \subseteq S \setminus J^\perp \subseteq S \setminus K^\perp$ (since $J \subseteq K$).

Assume, for a contradiction, that $y \in K$. Then $J \cup \{y\}$ is an irreducible subset of K' that contains J properly, because $y \in J^*$ and $y \neq s$. As J was chosen as an irreducible component of $(\langle K' \rangle, K')$, this is a contradiction. So $y \notin K$ and hence $y \in K^*$, since $y \notin K^\perp$. Since we are given that $\mathrm{o}(xs) = \infty$ for all $x \in K^*$, it follows that $\mathrm{o}(ys) = \infty$, contradicting the fact that y and s are adjacent in Π_{J^*}. □

Definition 13.3. Let (W, S) be a Coxeter system. An element $s \in S$ is called a *flexible generator* of (W, S) if the component of $S \setminus s^\infty$ containing s is spherical and contained in $(s^\infty)^\perp \cup \{s\}$. The *flexible factor* of a flexible generator s is the (spherical) component of $S \setminus s^\infty$ that contains s, and the *type* of a flexible generator is the type of its flexible factor.

If J is the flexible factor of a flexible generator s, and s is the i-th vertex of J (in the Bourbaki numbering of the Coxeter diagram), then we also say that s is a flexible generator of type (X, i) (where J is of type X).

Note that if J is a component of a Coxeter system (W, S) and J is spherical, then each $s \in J$ is a flexible generator of (W, S) and J is its flexible factor.

The following proposition is a reformulation of Definition 13.3.

Proposition 13.4. *Let (W, S) be a Coxeter system, let $s \in S$ and let $J \subseteq S$. Then s is a flexible generator of (W, S) and J is its flexible factor if and only if the following hold:*

(i) *J is spherical and irreducible;*
(ii) $S \setminus J \subseteq s^\perp \cup s^\infty$;
(iii) $S \setminus J \subseteq (J \setminus \{s\})^\perp$.

Proof. Assume first that s is a flexible generator and J its flexible factor.

Since J is a component of $S \setminus s^\infty$ and spherical, it is certainly spherical and irreducible. So (i) holds.

Since J is a component of $S \setminus s^\infty$ it follows that $S \setminus s^\infty \subseteq J^\perp \cup J$. But $J^\perp \subseteq s^\perp$ since $s \in J$, and so $S \setminus s^\infty \subseteq s^\perp \cup J$. So (ii) holds.

Since $J \subseteq (s^\infty)^\perp \cup \{s\}$ it follows that $J \setminus \{s\} \subseteq (s^\infty)^\perp$, or, equivalently, $s^\infty \subseteq (J \setminus \{s\})^\perp$. But $S \setminus s^\infty \subseteq J^\perp \cup J$, and so $S \setminus J \subseteq s^\infty \cup J^\perp \subseteq (J \setminus \{s\})^\perp$, since $J^\perp \subseteq (J \setminus \{s\})^\perp$ is obvious. Hence (iii) holds.

Now assume, conversely, that (i), (ii) and (iii) are satisfied. Since $s \notin s^\perp \cup s^\infty$, it follows from (ii) that $s \in J$, and since J is spherical it follows from this that $J \subseteq S \setminus s^\infty$. But it follows from (ii) and (iii) that $S \setminus J \subseteq (s^\perp \cap (J \setminus \{s\})^\perp) \cup s^\infty$, that is, $S \setminus J \subseteq J^\perp \cup s^\infty$. So $S \setminus s^\infty \subseteq J^\perp \cup J$, and so J is a union of components of $S \setminus s^\infty$. Since J is irreducible, it is a single component of $S \setminus s^\infty$.

To show that s is a flexible generator with J as its flexible factor, it remains to show that $J \subseteq (s^\infty)^\perp \cup \{s\}$. Since $J \subseteq S \setminus s^\infty$ it follows that $s^\infty \subseteq S \setminus J \subseteq (J \setminus \{s\})^\perp$, by (iii). So $J \setminus \{s\} \subseteq (s^\infty)^\perp$, which gives the required result. □

In view of Proposition 13.4, Proposition 13.2 can be reformulated as follows.

COROLLARY 13.5. *Let (W, S) be a twist rigid Coxeter system and $K \subseteq S$ an irreducible spherical subset, and let $s \in K$ be such that $K^* \subseteq s^\infty$. Then s is a flexible generator of (W, S) and K is the flexible factor of s.*

Proof. The assumption that $K^* \subseteq s^\infty$ means precisely that $\mathrm{o}(sx) = \infty$ for all $x \in K^*$. Hence we have $S \setminus K \subseteq s^\perp \cup s^\infty$ and $S \setminus K \subseteq t^\perp$ for all $t \in K \setminus \{s\}$ by Proposition 13.2. So conditions (ii) and (iii) of Proposition 13.4 are satisfied (with K in place of J), and since (i) is given the desired conclusion follows. □

The following definition simplifies the statement of Theorem 3.

DEFINITION 13.6. Let (W, S) be a Coxeter system. A *flexible pair of generators* of (W, S) is a pair $(s, t) \in S \times S$ such that $s \neq t$ and

(i) s is a right-angled element of S,
(ii) t is a flexible generator of (W, S),
(iii) $s^\infty = t^\infty$.

The *type* of the flexible pair (s, t) is the type of the flexible generator t.

Definitions 13.6, 7.6 and 7.5 immediately yield Proposition 13.7, in which types (D_n, n) and $(A_3, 3)$ should be regarded as including types $(D_n, n-1)$ and $(A_3, 1)$.

PROPOSITION 13.7. *Let (W, S) be a Coxeter system, and suppose that (s, t) a flexible pair of generators of (W, S) of type $I_2(n)$ or (D_n, n), for n odd and $n \geqslant 5$, or of type A_2 or $(A_3, 3)$. Then (s, t) is a proper blowing down pair for (W, S).*

13.2. *Some observations on twist rigid Coxeter systems*

LEMMA 13.8. *Let (W, S) be a twist rigid Coxeter system and $M \subseteq S$ an odd component of (W, S) such that $M^\infty \neq \emptyset$ and $C_0(M)$ is spherical. Then $M = \{s\}$ for some $s \in S$ such that s*

is a flexible generator of (W,S) *with* $C_0(M)$ *as its flexible factor. In particular,* s *is either of type* A_1 *or* C_2*, or else of type* (C_n, n) *or* $I_2(2n)$ *for some* $n \geqslant 3$*.*

Proof. Since $K := C_0(M)$ is the component of $S \setminus M^\infty$ that contains M, it follows that $K^* = M^\infty$. Thus $K^* \subseteq s^\infty$ holds for those $s \in K$ such that $s \in M$. Since we are given that (W,S) is twist rigid, it follows from Corollary 13.5 that each $s \in M$ is flexible with flexible factor K. So $K \setminus \{s\} \subseteq (S \setminus K)^\perp$, by Proposition 13.4 (iii), and since $M \subseteq C_0(M) = K$ it follows that $M \setminus \{s\} \subseteq (S \setminus C_0(M))^\perp \subseteq (M^\infty)^\perp$, since $M^\infty \subseteq S \setminus C_0(M)$. This gives $\mathrm{o}(tx) = 2$ for all $t \in M \setminus \{s\}$ and $x \in M^\infty$, although by definition $\mathrm{o}(tx) = \infty$ for all $t \in M$ and $x \in M^\infty$. Since $M^\infty \neq \emptyset$ this forces $M \setminus \{s\} = \emptyset$, and so $|M| = 1$. Thus $(\langle C_0(M)\rangle, C_0(M))$ is an irreducible spherical system in which $\{s\}$ is a singleton odd component. It follows that $C_0(M)$ is of type A_1 or $I_2(2n)$ (including $C_2 = I_2(4)$), or of type C_n with s as vertex number n. □

LEMMA 13.9. *Let* (W,S) *be a twist rigid Coxeter system,* $M \subseteq S$ *an odd component of* (W,S)*, and* $b \in C_0(M)$ *a* C_3*-neighbour of* M*. Suppose that* $c \in M$ *and* $a \in M$ *satisfy* $\mathrm{o}(bc) = 4$ *and* $\mathrm{o}(ac) = 3$*, and let* $K := \{a, c, b\}$*. Then* c *is a flexible generator of* (W,S) *of type* $(C_3, 2)$*, and* K *is the flexible factor of* c*.*

Proof. By Lemma 8.3 the element $a \in M$ such that $\mathrm{o}(ac) = 3$ is the only element of M with $\mathrm{o}(ac) \neq \infty$. Furthermore, Lemma 8.3 also says that $\{a, c, b\}$ is of type C_3.

If $e \in S \setminus K$ and $e \in M$ then $e \in M \setminus \{a, c\}$, and (N1) in Definition 8.2 yields $\mathrm{o}(ce) = \infty$. If $e \in S \setminus K$ and $e \notin M$ then $e \in S \setminus (M \cup \{b\})$, and (N2) in Definition 8.2 yields that either $\mathrm{o}(ce) = \infty$ or $e \in K^\perp$. So $S \setminus K \subseteq c^\infty \cup K^\perp$. That is, $K^* = S \setminus (K \cup K^\perp) \subseteq c^\infty$. By Corollary 13.5 it follows that c is a flexible generator of (W,S) and that K is its flexible factor. □

LEMMA 13.10. *Let* (W,S) *be a twist rigid Coxeter system and let* (z, a) *be a blowing down pair for* (W,S)*. Let* C *be the component of* $S \setminus z^\infty$ *that contains* a*, and let* $b := \rho_C a \rho_C$*. Then* (z, a) *is a proper blowing down pair for* (W,S)*. Furthermore, if* $t \in \{a, b\}$ *is such that* $z^\infty \subseteq t^\infty$*, then* (z, t) *is a flexible pair of generators of* (W,S)*, and* C *is the flexible factor of* t*.*

Proof. Since z is right-angled $J := z^\infty = S \setminus (\{z\} \cup z^\perp) = z^*$. So the graph Π_J is connected, as (W,S) is twist rigid. Therefore $z^\infty \subseteq a^\infty$ or $z^\infty \subseteq b^\infty$, by (iii) in Definition 7.5. Thus (z, a) is a proper blowing down pair for (W,S).

Let $t \in \{a, b\}$ be such that $z^\infty \subseteq t^\infty$. Note first that $S \setminus z^\infty \subseteq C \cup C^\perp$, since C is a component of $S \setminus z^\infty$. Since $t \in \{a, b\} \subseteq C$, and C is spherical, we see that $C \subseteq S \setminus t^\infty$. Furthermore, $t \in C$ also gives $C^\perp \subseteq t^\perp \subseteq S \setminus t^\infty$. Thus $S \setminus z^\infty \subseteq S \setminus t^\infty$, and so it follows that $z^\infty = t^\infty$.

As $S \setminus z^\infty \subseteq C \cup C^\perp$ we have $C^* = S \setminus (C \cup C^\perp) \subseteq z^\infty = t^\infty$. Since C is irreducible and spherical it follows from Corollary 13.5 that t is a flexible generator of (W,S) with flexible factor C. □

14. *Angle compatible Coxeter generating sets*

14.1. *Definitions and basic observations*

We recall that an edge of a Coxeter system (W,S) is an edge of its presentation diagram. That is, an edge is a set $\{s, t\} \subseteq S$ such that $s \neq t$ and the visible subgroup $\langle s, t\rangle$ is finite.

Definition 14.1. Let (W, S) be a Coxeter system. A subset X of W is called *S-compatible* if $X^w \subseteq S$ for some $w \in W$.

Thus $x \in W$ is a reflection of (W, S) if and only if $\{x\}$ is S-compatible. Moreover, if R is a Coxeter generating set for W then R is S-compatible if and only if $R^w = S$ for some $w \in W$, since no proper subset of S generates W.

Lemma 14.2. *Let (W, S) be a Coxeter system. If $R \subseteq S^W$ and $\langle R \rangle = W$, then $R^W = S^W$.*

Proof. Since $R \subseteq S^W$ it follows that $R^W \subseteq S^W$, and it suffices to prove that $S \subseteq R^W$. Let $s \in S$ and M the odd component of S containing s. It follows from the Coxeter presentation that there is a homomorphism $\varphi\colon W \to \{\pm 1\}$ with $\varphi(t) = -1$ for all $t \in M$ and $\varphi(t) = 1$ for all $t \in S \setminus M$. Clearly φ is surjective, since $\varphi(s) = -1$. Since $\langle R \rangle = W$ there must be some $r \in R$ with $\varphi(r) = -1$. Since $R \subseteq S^W$ there is a $w \in W$ such that $s' := w^{-1} r w \in S$, and since $\varphi(s') = \varphi(r) = -1$ it follows that $s' \in M$. So s is conjugate to s', and $s \in R^W$, as required. □

Lemma 14.3. *Let (W, S) be a Coxeter system and let $R \subseteq W$ be a Coxeter generating set for W. If each edge of (W, S) is R-compatible, then $S^W = R^W$ and each edge of R is S-compatible.*

Proof. Let $s \in S$. If s is not contained in an edge of (W, S), then $C_W(s) = \langle s \rangle$ by the main result of [**2**] (or by Lemma 3.2). By Proposition 3.8 every involution has the form $w^{-1} \rho_J w$ for some $w \in W$ and some $J \subseteq S$ of (-1)-type, and thus commutes with all the reflections $w^{-1} s w$ for $s \in J$. In particular, the centralizer of any involution x that is not a reflection contains $\langle x \rangle$ properly, because $|J| > 1$ if x is not a reflection. Hence $s \in R^W$ if s is not contained in any edge of (W, S). If s is contained in an edge $\{s, t\}$ of (W, S), then $s \in \{s, t\} \subseteq R^W$ by the hypothesis. So $s \in R^W$ in either case, and so $S^W = R^W$ by Lemma 14.2. The other assertion follows from Corollary A.4 in [**5**]. □

Definition 14.4. Two Coxeter generating sets R, S of a Coxeter group W are called *reflection equivalent* if $S^W = R^W$, and *angle compatible* if each edge of (W, S) is R-compatible.

Our aim is now to collect information about R-compatible edges. We first recall some well known facts about finite dihedral groups.

Lemma 14.5. *Let (W, S) be a Coxeter system of type $I_2(n)$, and let $S = \{s, t\}$.*

(i) *Suppose that $n \in \{2, 3, 4, 6\}$ and that $a, b \in S^W$ satisfy $W = \langle a, b \rangle$. Then there exists an element $w \in W$ such that $\{a, b\} = S^w$.*

(ii) *Suppose that $n = 5$ or $n \geqslant 7$. Then W has an automorphism α satisfying $\alpha(s) = s$ and $\alpha(t) \in S^W$, and $\alpha(S) \notin \{\, S^w \mid w \in W \,\}$.*

Proof. If $n = 5$ or $n \geqslant 7$ then there is a positive integer k such that $1 < k < n - 1$ and k is coprime to n. Hence if we define $t' = s(st)^k$ then t' is a reflection such that $st' = (st)^k$ has order n, and there is an automorphism α satisfying $\alpha(s) = s$ and $\alpha(t) = t'$. We leave it to the reader to complete the proof of (ii) by checking that $\alpha(S)$ is not conjugate to S, and also to prove (i). □

REMARK. The set $\{s, t'\}$ defined in the proof of Lemma 14.5 (ii) is a Coxeter generating set for W that is not angle compatible with $S = \{s, t\}$, since the unique edges of the two systems are not conjugate subsets of W. This example explains our terminology: in the standard geometrical representation of W the elements s and t are represented by reflections in lines that intersect at an angle of π/n, while s and t' are represented by reflections in lines that intersect at an angle of $k\pi/n$.

LEMMA 14.6. *Let (W, S) be a Coxeter system and $R \subseteq W$ a Coxeter generating set such that $S^W = R^W$. Let $J = \{s, t\}$ be an edge of (W, S). Then the following hold.*

(i) *There exists an edge K of (W, R) such that $\langle J\rangle^w = \langle K\rangle$ for some $w \in W$.*
(ii) *If $\mathrm{o}(st) \in \{2, 3, 4, 6\}$ then J is R-compatible.*

Proof. Since $\langle J\rangle$ is finite and $J \subseteq R^W$ it follows from Proposition 3.7 that there exist a spherical subset K of R and a $w \in W$ such that $|K| \leqslant |J|$ and $\langle J\rangle^w \subseteq \langle K\rangle$. Since $|J| = 2$ and $|\langle J\rangle| \geqslant 4$ it follows that $|K| = 2$, and so K is an edge of (W, R).

By a symmetrical argument there is an edge L of (W, S) and a $v \in W$ such that $\langle K\rangle^v \subseteq \langle L\rangle$, Thus $J^{wv} \subseteq \langle J\rangle^{wv} \subseteq \langle K\rangle^v \subseteq \langle L\rangle$, and as $|J| = |L| = 2$ equality must hold, by Lemma 3.3. Hence $\langle J\rangle^w = \langle K\rangle$.

Suppose now that $\mathrm{o}(st) \in \{2, 3, 4, 6\}$. By (i) there exist an edge K of (W, R) and a $w \in W$ such that $\langle J\rangle^w = \langle K\rangle$. Thus $\langle wKw^{-1}\rangle = \langle J\rangle$, and $wKw^{-1} \subseteq J^{\langle J\rangle}$ by Proposition 3.4. By Lemma 14.5 (i), it follows that there exists $v \in \langle J\rangle$ such that $J^v = wKw^{-1}$, and hence $J^{vw} = K$. Thus J is R-compatible. □

PROPOSITION 14.7. *Let (W, S) be a Coxeter system that is of type H_3 or H_4, or of type $I_2(n)$ for $n = 5$ or $n \geqslant 7$. Let $a \in S$ and $L := S \setminus \{a\}$. Then W has an automorphism α such that $\langle\alpha(L)\rangle = \langle L\rangle$ and $\alpha(S) \subseteq S^W$, and $\alpha(S) \notin \{\, S^w \mid w \in W \,\}$.*

Proof. If (W, S) is of type $I_2(5)$ or $I_2(n)$ for $n \geqslant 7$ then the claim follows from Lemma 14.5 (ii). Thus we are left with the case that (W, S) is of type H_3 or H_4.

By assertions (10) and (11) of [**10**, Theorem 31], there exists an automorphism β of W such that $\beta(S) \subseteq S^W$ and $\beta(S) \notin \{\, S^w \mid w \in W \,\}$. By Proposition 32 in [**10**] the group $\beta(\langle L\rangle)$ is a parabolic subgroup of W. Thus there exist $v \in W$ and $J \subseteq S$ such that $(\gamma_v\beta)(\langle L\rangle) = \langle J\rangle$, where γ_v is the inner automorphism of W corresponding to v. As $\langle J\rangle$ is isomorphic to $\langle L\rangle$, and $\langle L\rangle$ is a maximal proper subset of S, it follows that $L = J$, as is readily checked from the H_3 and H_4 diagrams. So the automorphism $\alpha := \gamma_v\beta$ has the required properties. □

14.2. *Angles in Coxeter systems*

We collect here the results from [**16**] that we need. It is appropriate to point out that the notation and terminology we use differs slightly from that used in [**16**]. In particular, if S and R are reflection equivalent Coxeter generating sets for the group W, then an R-compatible edge of (W, S) was called *sharp-angled with respect to* R in [**16**]. Furthermore, if $K \subseteq S$ then we shall use the notation $K^{(\infty)}$ for the subset of S that was denoted by K^∞ in [**16**].

NOTATION. If (W, S) is a Coxeter system and $K \subseteq S$, then $K^{(\infty)} := \left(\bigcup_{s\in K} s^\infty\right) \setminus K$.

Recall that by definition $K^* = S \setminus (K \cup K^\perp)$. Thus $K^{(\infty)} \subseteq K^*$.

Definition 14.8. Let (W, S) be a Coxeter system. If K is a subset of S then K is called *2-spherical* if $\mathrm{o}(st) \neq \infty$ for all s, $t \in K$, and is called *strongly 2-spherical* if it is 2-spherical, irreducible and not spherical.

Proposition 14.9. *Let (W, S) be a Coxeter system, $J := \{r, s\}$ an edge of (W, S), and $R \subseteq S^W$ another Coxeter generating set for W. Then the following hold.*

(i) *If there is a strongly 2-spherical subset K of S containing J then J is R-compatible.*

(ii) *Suppose that J is not R-compatible, that $K \subseteq S$ is an irreducible, spherical subset of S containing J, and that L is a connected component of Π_M, where $M = K^{(\infty)}$. Then there is an $a \in K$ such that $L \subseteq a^\infty$.*

Proof. Since (i) follows immediately from [**16**, Theorem 5.9], we have only to prove (ii). So we assume that J is not R-compatible, and K and L satisfy the hypotheses of (ii).

By (ii) in Lemma 14.6 we have $\mathrm{o}(rs) \geqslant 5$. Since $K \subseteq S$ is spherical and irreducible and contains r and s, it follows from the classification of spherical systems that either $|K| = 2$, in which case $K = \{r, s\} = J$, or else K is of type H_3 or H_4.

By [**16**, Proposition 8.4], the subset J is a Δ-edge of S, in the sense of [**16**, Definition 8.3]. In the language of [**16**], the claim in (ii) is that K is a flexible subset of S. (For the definition of this see [**16**, Subsection 2.6].)

If $K = J$ and $\mathrm{o}(rs) > 5$ then our claim follows from Proposition 6.1 of [**16**]. (For the definition of a Θ-edge see [**16**, Definition 4.5].)

If $K = J$ and $\mathrm{o}(rs) = 5$ then our claim follows from Lemma 9.1 of [**16**].

If K is of type H_3 or H_4 then our claim follows from Proposition 9.3 of [**16**] or Proposition 9.7 of [**16**] (respectively). □

14.3. *Angles in twist rigid Coxeter systems*

Lemma 14.10. *Let (W, S) be a Coxeter system, let $R \subseteq S^W$ be a Coxeter generating set of W, and let $J = \{r, s\} \subseteq S$ be an edge that is not R-compatible. Let K be an irreducible spherical subset of S that contains J and is not contained properly in any irreducible spherical subset of S. Then the following hold:*

(i) $K^{(\infty)} = K^*$;

(ii) *if (W, S) is twist rigid, then there is an $a \in K$ such that a is flexible generator of (W, S) with flexible factor K.*

Proof. Let $x \in K^*$. As K is irreducible, it follows that $L := K \cup \{x\}$ is irreducible, and as K is a maximal irreducible spherical subset of S it follows that L is not spherical. Therefore either $x \in K^{(\infty)}$ or L is strongly 2-spherical. As L contains J the latter is not possible in view of Proposition 14.9 (i). Thus $K^* \subseteq K^{(\infty)}$, and we conclude that $K^* = K^{(\infty)}$, since the reverse inclusion is clear from the definitions.

Assume now that (W, S) is twist rigid. As K is irreducible and spherical, it follows from the definition of twist rigidity that the graph Π_{K^*} is connected. Hence, by (i), the graph $\Pi_{K^{(\infty)}}$ has only one connected component. Thus it follows from Proposition 14.9 (ii) that there exists an $a \in K$ such that $K^* \subseteq a^\infty$, and an application of Corollary 13.5 completes the proof. □

Proposition 14.11. *Let (W, S) be a twist rigid Coxeter system and $R \subseteq S^W$ a Coxeter generating set for W. Let $J = \{r, s\} \subseteq S$ be an edge that is not R-compatible. Then J is contained in a unique maximal irreducible spherical subset K of S. Moreover, some element $a \in K$ is a flexible generator of (W, S) with flexible factor K.*

Proof. Choose $K \subseteq S$ to be a maximal irreducible spherical subset containing J. By Lemma 14.10 (ii), some $a \in K$ is a flexible generator of (W, S) with flexible factor K.

Let $c \in K$ and let $d \in S$ be such that $2 < \mathrm{o}(cd) < \infty$. Since $S \setminus K \subseteq c^\infty \cup c^\perp$ it follows that d is also in K. This shows $L \subseteq K$ for each irreducible spherical subset L of S that contains J. Thus K is unique. □

15. *Proof of Theorem 2*

PROPOSITION 15.1. *Let (W, S) be a Coxeter system, let $K \subseteq S$, and let $s \in K$ be a flexible generator of (W, S) with flexible factor K. Suppose that α is an automorphism of $\langle K\rangle$ such that $\alpha(K) \subseteq K^{\langle K\rangle}$ and $\alpha(K \setminus \{s\}) \subseteq \langle K \setminus \{s\}\rangle$. Then there exists an automorphism β of W such that $\beta\,|_{\langle K\rangle} = \alpha$ and $\beta(r) = r$ for all $r \in S \setminus K$. Moreover, $\beta(S)^W = S^W$, and if $\alpha(K)$ is not conjugate to K in $\langle K\rangle$ then $\beta(S)$ is not conjugate to S in W.*

Proof. Let $f\colon S \to W$ be defined by $f(r) = \alpha(r)$ for all $r \in K$ and $f(r) = r$ for all $r \in S \setminus K$. We show that W has an endomorphism β such that $\beta(r) = f(r)$ for all $r \in S$. Since S is a Coxeter generating set for W, it suffices to check that $(f(r)f(t))^{\mathrm{o}(rt)} = 1$ for all r, $t \in S$ such that $\mathrm{o}(rt) \neq \infty$. Furthermore, if r and t are both in K or both in $S \setminus K$ then it is obvious that $f(r)f(t)$ and rt have the same order. Hence we may assume that $r \in K$ and $t \in S \setminus K$, so that $f(r) = \alpha(r)$ and $f(t) = t$, and our task is to show that if $\mathrm{o}(rt) \neq \infty$ then $(\alpha(r)t)^{\mathrm{o}(rt)} = 1$.

Let $r \in K$ and $t \in S \setminus K$ be such that $\mathrm{o}(rt) \neq \infty$. Since K is the flexible factor of s, it follows from Definition 13.3 and Proposition 13.4 that $S \setminus K \subseteq K^\perp \cup s^\infty$ and that $K \setminus \{s\} \subseteq (s^\infty)^\perp$. Suppose first that $t \notin s^\infty$, and observe that $\mathrm{o}(rt) = 2$, since $t \in K^\perp$ and $r \in K$. Furthermore, t commutes with all elements of $\langle K\rangle$, since it commutes with all elements of K. In particular, t commutes with $\alpha(r)$, since $r \in K$ and α is an automorphism of $\langle K\rangle$. So $(\alpha(r)t)^2 = 1$, as required. On the other hand, suppose that $t \in s^\infty$. Then $r \neq s$, since $\mathrm{o}(rt) \neq \infty$. So $r \in K \setminus \{s\} \subseteq (s^\infty)^\perp$, and again it follows that $\mathrm{o}(rt) = 2$. Furthermore, t commutes with all elements of $\langle K \setminus \{s\}\rangle$, since it commutes with all elements of $(s^\infty)^\perp$, and, in particular, it commutes with $\alpha(r)$ since $\alpha(K \setminus \{s\}) \subseteq \langle K \setminus \{s\}\rangle$. So again $(\alpha(r)t)^2 = 1$, as required. Hence W has an endomorphism β extending the map f.

It is clear that $\beta|_{\langle K\rangle} = \alpha$. As $\langle K\rangle$ is finite group, there exists an $m \geqslant 1$ such that α^m is the identity on $\langle K\rangle$. Thus $\beta^m(r) = r$ for all $r \in K$, and hence for all $r \in S$. So β^m is the identity on W, and therefore β is an automorphism. As $\beta(S) = \beta(S \setminus K) \cup \beta(K) = S \setminus K \cup \alpha(K)$ and $\alpha(K) \subseteq K^{\langle K\rangle} \subseteq S^W$, we have $\beta(S) \subseteq S^W$. Thus $\beta(S)^W = S^W$, by Lemma 14.2. For the final assertion, suppose that $\beta(S)$ is conjugate to S, so that $\beta(S) = S^w$ for some $w \in W$. Then $\beta(K) = J^w$ for some $J \subseteq S$, and $J^w = \alpha(K) \subseteq \langle K\rangle$. By Proposition 3.3 there exists an element $v \in \langle K\rangle$ such that $\alpha(K)^v = J^{wv} \subseteq K$, and since $|\alpha(K)| = |K|$ it follows that $\alpha(K)$ is conjugate to K in $\langle K\rangle$. □

COROLLARY 15.2. *Let (W, S) be a Coxeter system, and suppose that (W, S) has a flexible generator s of type H_3 or H_4, or $I_2(n)$ for $n = 5$ or $n \geqslant 7$. Then W has an automorphism β such that $\beta(S) \subseteq S^W$ and $\beta(S) \notin \{\, S^w \mid w \in W \,\}$. Thus (W, S) is not strongly reflection rigid.*

Proof. Let K be the flexible factor of s, and let $J := K \setminus \{s\}$. By Proposition 14.7 there is an automorphism α of $\langle K\rangle$ satisfying $\langle\alpha(J)\rangle = \langle J\rangle$ and $\alpha(K) \subseteq K^{\langle K\rangle}$, and $\alpha(K) \notin \{\, K^v \mid v \in \langle K\rangle \,\}$. Applying Proposition 15.1 completes the proof. □

The following result of Caprace and Przytycki is Theorem 1.1 in [**5**].

Proposition 15.3. *Let R and S be angle compatible Coxeter generating sets for a group W. If (W,S) is twist rigid, then $R = S^w$ for some $w \in W$.*

Proof of Theorem 2. Let (W,S) be a twist rigid Coxeter system of finite rank.

If (W,S) has a flexible generator of type H_3 or H_4, or $I_2(n)$ for $n = 5$ or $n \geqslant 7$, then (W,S) is not strongly reflection rigid, by Corollary 15.2. Thus if (W,S) is strongly reflection rigid then (W,S) satisfies conditions (i) and (ii) of Theorem 2.

Conversely, suppose that (W,S) is not strongly reflection rigid. Since it is twist rigid, by hypothesis, it follows from Proposition 15.3 that W has a Coxeter generating set $R \subseteq S^W$ that is not angle compatible with S. So there is an edge $J = \{s,t\}$ of (W,S) such that J is not R-compatible. By Proposition 14.11 it follows that there is a unique maximal irreducible spherical subset K of S that contains J; moreover, there is an $a \in K$ such that a is a flexible generator of (W,S) and K is its flexible factor. As J is not R-compatible it follows from Lemma 14.6 (ii) that $\mathrm{o}(st) = 5$ or $\mathrm{o}(st) \geqslant 7$. As K is an irreducible spherical subset of S containing J it follows that K is of type H_3 or H_4, or of type $I_2(n)$ for $n = 5$ or $n \geqslant 7$. So either condition (i) of Theorem 2 is not satisfied, or else condition (ii) of Theorem 2 is not satisfied. □

16. *Proof of Theorem 3*

Lemma 16.1. *Let (W,S) be a Coxeter system. Then (W,S) is not strongly rigid if any of the following hold.*

- (PT) *The system (W,S) admits a proper twist.*
- (MC) *The generating set S has a mutable odd component.*
- (i) *There is a flexible generator of type H_3 or H_4;*
- (ii) *There is a flexible generator of type $I_2(n)$ for some $n \geqslant 5$;*
- (iii) *There is a flexible generator of type (C_n, n) for some $n \geqslant 3$;*
- (iv) *There is a flexible generator of type $(C_3, 2)$;*
- (v) *There is a flexible pair of type (D_n, n) or $(D_n, n-1)$ with n odd and $n \geqslant 5$;*
- (vi) *There is a flexible pair of type A_2 or of type $(A_3, 1)$ or $(A_3, 3)$;*
- (vii) *There is an irreducible component of type A_5.*
- (viii) *The system (W,S) is of type F_4, E_8 or D_{2n} for some $n \geqslant 2$.*

Proof. If (W,S) admits a proper twist it is not strongly rigid, by Propositions 4.5 and 4.6.

If S has a mutable odd component M, then the elements of M are not intrinsic reflections of W, by Proposition 5.3. Thus (W,S) is not strongly rigid in this case.

If there exists a flexible generator of type H_3 or H_4, or $I_2(n)$ for $n = 5$ or $n \geqslant 7$, then (W,S) is not strongly reflection rigid by Corollary 15.2. Thus (W,S) is not strongly rigid. If there exists a flexible generator s of type $I_2(6)$, then s is a blowing up generator of (W,S), and by Corollary 7.2 it follows that s is not an intrinsic reflection of (W,S). Thus (W,S) is not strongly rigid in this case as well. Thus, if either (i) or (ii) hold, then (W,S) is not strongly rigid.

Suppose that s is a flexible generator of type (C_n, n), with $n \geqslant 3$. Then $M := \{s\}$ is the odd component of (W,S) containing s, and $C_0(M)$ is the flexible factor of s. As $C_0(M)$ is of type C_n with $n \geqslant 3$, it follows from Theorem 1 (ii) that s is not an intrinsic reflection of (W,S). Thus if (iii) holds then (W,S) is not strongly rigid.

Suppose that s is a flexible generator of type $(C_3, 2)$, and let $K = \{a, s, b\}$ be the flexible factor of s, with $\mathrm{o}(as) = 3$ and $\mathrm{o}(sb) = 4$. Then $M = \{a, s\}$ is the odd component of (W,S) containing s, and b is a C_3-neighbour of M. Note also that $s^\infty \subseteq C_0(M)$, since $s^\infty \subseteq (K \setminus \{s\})^\perp \subseteq a^\perp$. If $s^\infty = \emptyset$ then K is a component of S, and it follows from Theorem 1 (ii) that s is not an intrinsic reflection of (W,S). If $s^\infty \neq \emptyset$ then $C_0(M)$ is non-spherical, and Theorem 1 (iv) tells us that s is not an intrinsic reflection of (W, S). So, in either case, (W, S) is not strongly rigid. Thus if (iv) holds then (W, S) is not strongly rigid.

Suppose that (z, s) is a flexible pair of type A_2, or of type $(A_3, 3)$ or $(A_3, 1)$, or of type (D_n, n) or $(D_n, n-1)$ for some odd number $n \geqslant 5$. Then (z, s) is a blowing down pair for (W, S), by Proposition 13.7, and therefore the generator z is not an intrinsic reflection of (W, S), by Theorem 1 (i). Thus if (v) or (vi) hold then (W, S) is not strongly rigid.

Suppose that there exists an irreducible component J of (W, S) of type A_5. Then J is an odd component of (W, S), and we have $J = C_0(J)$. Hence $s \in J$ is not an intrinsic reflection of $\langle J \rangle$, by Theorem 1 (iii). Thus if (vii) holds then W is not strongly rigid.

If (W, S) is of type F_4, E_8 or D_{2n} for some $n \geqslant 2$, then no element of S is an intrinsic reflection of W, by Proposition 6.5. Thus if (viii) holds then (W, S) is not strongly rigid. □

Proof of Theorem 3. It follows from Lemma 16.1 that the conditions (TR), (NM) and (i)–(viii) of Theorem 3 must all hold if (W, S) is strongly rigid, and our remaining task is to show that if they do all hold then (W, S) is indeed strongly rigid. Accordingly, we assume throughout this proof that (W, S) is a twist rigid Coxeter system that has no mutable odd components and satisfies conditions (i)–(viii) of Theorem 3.

As (W, S) is twist rigid and satisfies (i) and (ii), it follows from Theorem 2 that (W, S) is strongly reflection rigid. Thus it suffices to show that each $s \in S$ is an intrinsic reflection of W.

Let $s \in S$ and let M be the odd component of S containing s. Since S has no mutable odd components, M is not mutable. Our aim is to show that s is an intrinsic reflection of W.

We distinguish four cases.

Case 1. Suppose that $|C_0(M)| = 1$, that is, s is a right-angled element of S.

Assume, for a contradiction, that s is not an intrinsic reflection of (W, S). By Theorem 1 (i), there exists an $a \in S$ such that (s, a) is a blowing down pair for (W, S). As (W, S) is twist rigid by hypothesis, it follows by Lemma 13.10 that (s, a) is a flexible pair. By Definitions 7.5 and 7.6 the flexible pair (s, a) must be of type (D_n, n) (or $(D_n, n-1)$) or $I_2(n)$ for some odd number $n \geqslant 5$, or of type $(A_3, 3)$ (or $(A_3, 1)$) or A_2. If it is of type (D_n, n) or $(D_n, n-1)$ for $n \geqslant 5$ then (W, S) does not satisfy condition (v) of Theorem 3, contrary to our hypotheses. Similarly, if it is of type A_2 or of type $(A_3, 3)$ or $(A_3, 1)$ then condition (vi) of Theorem 3 is not satisfied, contrary to our hypotheses. If (s, a) is a flexible pair of type $I_2(n)$ for some $n \geqslant 5$, then a is a flexible generator of type $I_2(n)$ with $n \geqslant 5$, and condition (ii) of Theorem 3 is not satisfied, contrary to our hypotheses. Since all cases are impossible, it follows that s is an intrinsic reflection of W.

Case 2. Suppose that $C_0(M)$ is spherical of rank at least 2, and $M^\infty = \emptyset$.

By their definitions, $E(M) = S \setminus M^\infty$, and $C_0(M)$ is a component of $E(M)$. Thus, by the hypotheses of this case, $C_0(M)$ is a spherical component of S of rank at least 2. Put $J := C_0(M)$, and suppose first that J is of (-1)-type.

Since J is a spherical component of S, it is immediate from Definition 13.3 that every element of J is a flexible generator of (W, S). Since (i), (ii) and (iii) of Theorem 3 are assumed to hold, J is not of type H_3, H_4, $I_2(n)$ with $n \geqslant 5$, or C_n with $n \geqslant 3$. Since J is irreducible, of rank at least 2, and of (-1)-type, the remaining possibilities are C_2, F_4, E_7, E_8, and D_{2n} for $n \geqslant 2$.

Our aim is to show that s is an intrinsic reflection of W, and this will follow from (ii) of Theorem 1 if J is of type C_2 or E_7. Thus it will suffice to show that J cannot be of type F_4, E_8, or D_{2n} for $n \geqslant 2$.

Suppose that J is of type F_4, E_8, or D_{2n} for $n \geqslant 2$. Suppose, for a contradiction, that $J \neq S$, and let $r \in S \setminus J$. Let M' be the odd component of S containing r. Since $J \subseteq r^\perp \subseteq E(M')$ it follows that J is a component of $E(M')$ (since it is a component of S), and since $r \notin J$ it follows that J is an M'-subsidiary component of $E(M')$. Thus $E(M')$ has an M'-subsidiary component of (-1)-type, and hence M' is mutable, by definition. This contradicts the hypothesis that (W, S) has no mutable odd components.

Thus $S = J$, and so (W, S) is an irreducible spherical system of type F_4, E_8, or D_{2n} for $n \geqslant 2$. This contradicts our hypothesis that condition (viii) of Theorem 3 holds. So we have shown that if J is of (-1)-type then it must be of type C_2 or E_7, and hence s must be an intrinsic reflection of W, as required.

If J is not of (-1)-type then the required conclusion that s is an intrinsic reflection of W follows immediately from Theorem 1 (iii), since our hypothesis that condition (vii) of Theorem 3 is satisfied ensures that J is not of type A_5.

Case 3. Suppose that $C_0(M)$ is spherical of rank at least 2, and $M^\infty \neq \emptyset$.

As (W, S) is twist rigid, it follows from Lemma 13.8 that $M = \{s\}$ and that s is a flexible generator of (W, S) whose flexible factor is $C_0(M)$. As $|C_0(M)| > 1$, it follows that s is of type $I_2(2n)$ for some $n \geqslant 2$ (including $C_2 = I_2(4)$) or of type (C_n, n) for some $n \geqslant 3$. In view of condition (iii) the latter case is not possible, and in view of condition (ii) the former case is not possible with $n > 2$. So s is flexible generator of type C_2. It follows that $C_0(M)$ is of type C_2, and as M is not mutable it follows from Theorem 1 (ii) that s is an intrinsic reflection of W.

Case 4. Suppose that $C_0(M)$ is non-spherical.

Suppose that M has a C_3-neighbour. As (W, S) is twist rigid, it follows from Lemma 13.9 that there is a flexible generator c of (W, S) of type $(C_3, 2)$, contradicting the hypothesis that condition (iv) of Theorem 3 holds. Thus M has no C_3-neighbours, and, as M is not mutable, it follows from Theorem 1 (iv) that s is an intrinsic reflection of (W, S).

Since the four cases above are exhaustive, this completes the proof that if conditions (TR), (NM) and (i)–(viii) are satisfied then all elements of S are intrinsic reflections of W. This in turn completes the proof of Theorem 3. □

17. *Concluding remarks on strongly rigid Coxeter systems*

Our Theorem 3 provides a characterization of strongly rigid Coxeter systems in terms of their diagrams. In this final section we discuss several results about strong rigidity. For ease of reference we record those results that concern the most frequently discussed classes of Coxeter systems. Most of these results were known before, and we mention the relevant references in these cases. Furthermore, we comment on the logical dependence of some of these results. In this section all Coxeter systems are assumed to have finite rank.

17.1. *Spherical Coxeter systems*

The following proposition is a consequence of Theorem 3. Of course, it could be proved at much lesser cost by deducing it directly from the results in Section 6. We also remark that the irreducibility assumption in the proposition is not a severe restriction, in view of the discussion in the next subsection. It is merely convenient for obtaining a concise statement.

PROPOSITION 17.1. *Let (W, S) be an irreducible spherical Coxeter system. Then (W, S) is strongly rigid if and only if one of the following holds:*

(i) *(W, S) is of type A_n for some $n \neq 5$;*
(ii) *(W, S) is of type D_n for some odd $n \geqslant 5$;*
(iii) *(W, S) is of type C_2, E_6 or E_7.*

17.2. *Reducible Coxeter systems*

Let (W, S) be a reducible Coxeter system and suppose that $J \subset S$ is an irreducible component of (-1)-type. Let $s \in S \setminus J$ and let M be the odd component of (W, S) containing s. Then J is a subsidiary component of $E(M)$ of (-1)-type and therefore s is not an intrinsic reflection of W by Proposition 5.3. Combining these considerations with Proposition 17.1 one obtains the following.

PROPOSITION 17.2. *Let (W, S) be a Coxeter system such that the centre of W is non-trivial. Then (W, S) is strongly rigid if and only if (W, S) is of type A_1, C_2 or E_7.*

Let (W, S) be a Coxeter system. For investigating strong rigidity of (W, S) it is rather harmless to assume that W has trivial centre, in view the previous proposition. Suppose now that W has trivial centre. Then, by the main result of [**23**], the decomposition of (W, S) into its irreducible components is a Remak decomposition of W, and by Proposition 8 in [**9**] the Remak decomposition is unique. Thus we obtain the following.

PROPOSITION 17.3. *Let (W, S) be a Coxeter system such that the centre of W is trivial. Then (W, S) is strongly rigid if and only if each of its irreducible components is strongly rigid.*

17.3. *Strongly 2-spherical Coxeter systems*

A Coxeter system (W, S) is called 2-spherical if the order of st is finite for all $s, t \in S$. It is called *strongly 2-spherical* if it is 2-spherical and irreducible, and W is an infinite group.

PROPOSITION 17.4. *A strongly 2-spherical Coxeter system is strongly rigid.*

REMARK. The proposition follows by combining main results of [**4**] and [**11**]. In fact, these are the references where the result appears in the literature for the first time. A more conceptual proof of the result is given in [**6**] and we comment on this in more detail in the subsection on bipolar Coxeter groups below. Of course, the result would also follow from our Theorem 3. But, as we already pointed out in Subsection 2.4, Proposition 17.4 is used in the proof of Theorem 3.

17.4. *Right-angled Coxeter systems*

A Coxeter system (W, S) is called *right-angled* if $\mathrm{o}(st) \in \{1, 2, \infty\}$ for all s, $t \in S$. For $s \in S$ let $s^{\perp} := \{\, t \in S \mid \mathrm{o}(st) = 2 \,\}$ and $s^{\infty} := \{\, t \in S \mid \mathrm{o}(st) = \infty \,\}$.

PROPOSITION 17.5. *A right-angled Coxeter system (W, S) is strongly rigid if and only if the following conditions are both satisfied for each $s \in S$.*

(i) *For every $t \in s^{\perp}$ there exists $u \in s^{\perp}$ such that $o(tu) = \infty$.*
(ii) *For all t, $u \in s^{\infty}$ there exist a positive integer k and $t = t_0, t_1, \ldots, t_k = u$ in s^{∞} such that $t_{i-1}t_i = t_it_{i-1}$ for all $i \in \{1, 2, \ldots, k\}$.*

This follows from [**3**, Theorem 4.10].

17.5. *Affine and compact hyperbolic Coxeter systems*

For each positive integer n let E^n denote n-dimensional Euclidean space and let H^n denote n-dimensional hyperbolic space.

Definition 17.6. A Coxeter system (W, S) is called *affine* if there is a positive integer n such that W embeds in $\mathrm{Isom}(E^n)$ as a discrete and cocompact reflection subgroup, with the elements of S as reflections. It is called *compact hyperbolic* if there is a positive integer n such that W embeds in $\mathrm{Isom}(H^n)$ as a discrete and cocompact reflection subgroup, with the elements of S as reflections.

Proposition 17.7. *Let (W, S) be an affine or a compact hyperbolic Coxeter system. Then (W, S) is strongly rigid.*

Remark. The proposition is a consequence of the Main Theorem stated in the introduction of [**7**]. It asserts that any two Coxeter generating sets of a Coxeter group W that admits a proper and cocompact action on a contractible manifold are conjugate in W. An alternative proof of the Main Theorem in [**7**] is given in [**6**].

There is a well known classification of affine Coxeter systems in terms of their diagrams. Using this classification, it follows from our Theorem 3 that affine Coxeter systems are strongly rigid. In the compact hyperbolic case the situation is different. It seems that a classification of all compact hyperbolic Coxeter systems in terms of their diagrams is out of reach. Thus, in the compact hyperbolic case Proposition 17.7 cannot deduced from our Theorem 3 in this way.

17.6. *Bipolar Coxeter systems*

Bipolar Coxeter systems have been introduced by Caprace and Przytycki in [**6**]. They call a Coxeter system (W, S) bipolar if its Cayley graph satisfies a certain geometric condition. The following proposition summarizes some of the results in [**6**]. It underlines in particular that bipolar Coxeter systems are most relevant in the context of strong rigidity.

Proposition 17.8. *Let (W, S) be a Coxeter system. Then the following hold.*

(i) *If (W, S) is bipolar, then (W, S) is strongly rigid.*
(ii) *If (W, S) is strongly 2-spherical, then (W, S) is bipolar.*
(iii) *If W admits a proper and cocompact action on a contractible manifold, then (W, S) is bipolar.*

We make the following remarks on bipolar Coxeter systems and strong rigidity.

(1) As we have already mentioned, the results in [**6**] provide alternative proofs for Propositions 17.4 and 17.7. We can now make this more precise: combining assertions (i) and (ii) of Proposition 17.8 yields Proposition 17.4, while combining assertions (i) and (iii) of Proposition 17.8 yields the Main Theorem of [**7**], and hence also Proposition 17.7.

(2) A further major result on bipolar Coxeter systems is Theorem 1.2 in [**6**]. It provides a characterization of bipolar Coxeter systems in terms of their diagrams. It is not hard to verify that the conditions characterizing bipolar Coxeter systems are stronger than the conditions in our Theorem 3 that characterize strongly rigid Coxeter systems. Thus, assertion (i) of Proposition 17.8 follow by combining our Theorem 3 with Theorem 1.2 in [**6**].

(3) As already pointed out at the end of the introduction of [**6**], the converse of assertion (i) in Proposition 17.8 does not hold. It is not too hard to produce examples of Coxeter systems that do satisfy Conditions (TR), (MN) and (i)–(viii) of our Theorem 3 but do not satisfy conditions (a), (b) and (c) of Theorem 1.2 in [**6**]. One such example is Example $(**)$ given on page 37 of [**6**]. (Note, however, that Example $(*)$ given on page 37 of [**6**] is not strongly rigid.) It would be interesting to find a geometric condition for the Cayley graph of a Coxeter system that is weaker than bipolarity, but still implies strong rigidity.

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