Bicolor-eliminable graphs and free multiplicities on the braid arrangement

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Abstract

We define specific multiplicities on the braid arrangement by using edge-bicolored graphs. To consider their freeness, we introduce the notion of bicolor-eliminable graphs as a generalization of Stanley's classification theory of free graphic arrangements by chordal graphs. This generalization gives us a complete classification of the free multiplicities defined above. As an application, we prove one direction of a conjecture of Athanasiadis on the characterization of the freeness of the deformation of the braid arrangement in terms of directed graphs.

0 Introduction

Let $V = V^\ell$ be an ℓ-dimensional vector space over a field $\mathbb{K}$ of characteristic zero, $\{x_1, \ldots, x_\ell\}$ a basis for the dual vector space V^* and $S := \mathrm{Sym}(V^*) \simeq \mathbb{K}[x_1, \ldots, x_\ell]$. Let $\mathrm{Der}_\mathbb{K}(S)$ denote the S-module of $\mathbb{K}$-linear derivations of S, i.e., $\mathrm{Der}_\mathbb{K}(S) = \bigoplus_{i=1}^{\ell} S \cdot \partial_{x_i}$. A non-zero element $\theta = \sum_{i=1}^{\ell} f_i \partial_{x_i} \in \mathrm{Der}_\mathbb{K}(S)$ is *homogeneous of degree* p if f_i is zero or homogeneous of degree p for each i.

A *hyperplane arrangement* $\mathcal{A}$ (or simply an *arrangement*) is a finite collection of affine hyperplanes in V. If each hyperplane in $\mathcal{A}$ contains the origin, we say that $\mathcal{A}$ is *central*. In this article we assume that all arrangements are central unless otherwise specified. A *multiplicity* m on an arrangement $\mathcal{A}$ is a map $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ and a pair $(\mathcal{A}, m)$ is called a *multiarrangement*. Let $|m|$ denote the sum of the multiplicities $\sum_{H \in \mathcal{A}} m(H)$. When $m \equiv 1$, $(\mathcal{A}, m)$ is the same as the hyperplane arrangement $\mathcal{A}$ and sometimes called a *simple*

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arrangement. For each hyperplane $H \in \mathcal{A}$ fix a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. The first main object in this article is the *logarithmic derivation module* $D(\mathcal{A}, m)$ of $(\mathcal{A}, m)$ defined by

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) | \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \text{ (for all } H \in \mathcal{A})\}.$$

A multiarrangement $(\mathcal{A}, m)$ is *free* if $D(\mathcal{A}, m)$ is a free S-module of rank ℓ. If $(\mathcal{A}, m)$ is free, then there exists a homogeneous free basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A}, m)$. Then we define the *exponents* of a free multiarrangement $(\mathcal{A}, m)$ by $\exp(\mathcal{A}, m) := (\deg(\theta_1), \ldots, \deg(\theta_\ell))$. The exponents are independent of a choice of a basis. When $m \equiv 1$, the logarithmic derivation module and exponents are denoted by $D(\mathcal{A})$ and $\exp(\mathcal{A})$. When we fix a simple arrangement $\mathcal{A}$, we say that a multiplicity m on $\mathcal{A}$ is *free* (resp. *non-free*) if a multiarrangement $(\mathcal{A}, m)$ is free (resp. non-free).

A fundamental object of study in hyperplane arrangements is the arrangement of all reflecting hyperplanes of a Coxeter group, called a *Coxeter arrangement.* The study of the logarithmic derivation module for a Coxeter arrangement and its freeness were initiated by K. Saito in [17], developed in [18], and promoted by Solomon-Terao in [19], Terao in [22] and many other authors. In particular, Yoshinaga proved in [25] and [26] that the freeness of an arrangement is closely related to the canonical restricted multiarrangement defined by Ziegler in [28]. Hence the freeness of multiarrangements is now a very important subject of research.

Recently, some results were developed in [5] and [6] to study $D(\mathcal{A}, m)$ for general multiarrangements. Also, some results concerning free multiplicities on Coxeter arrangements have been found, e.g., see [3], [7] and [27]. In this article we generalize the study of free multiplicities on the braid arrangement.

A *braid arrangement* $\mathcal{A}_\ell$, or the *Coxeter arrangement of type* A_ℓ is defined as $\{H_{ij} := \{x_i - x_j = 0\} | 1 \leq i, j \leq \ell + 1, i \neq j\}$ in $V = V^{\ell+1}$. By using the primitive derivation introduced in [17], free multiplicities on Coxeter arrangements are studied by Solomon-Terao [19], Terao [22], Yoshinaga [24], and the first author and Yoshinaga [7]. Combining these results, we have a characterization of the freeness of quasi-constant multiplicities m on a Coxeter arrangement, i.e., multiplicities such that $\max_{H, H' \in \mathcal{A}} |m(H) - m(H')| \leq 1$. However, it is known that if $\max_{H, H' \in \mathcal{A}} |m(H) - m(H')| = 2$ then the same method using the primitive derivation does not work. Also, to determine explicitly which multiplicity makes $(\mathcal{A}, m)$ free is a difficult problem. Our aim is to consider these multiplicities on the braid arrangement and classify their freeness completely. In fact, we consider every multiplicity m such that $|2k - m(H_{ij})| \leq 1$ for some $k \in \mathbb{Z}_{>0}$ since, as shown in [7], a mysterious and interesting symmetry of the freeness and duality of exponents exists for these kinds of multiplicities m.

To state the main theorem, let us introduce some notation. Let $\mathcal{A}$ be the braid arrangement in $V^{\ell+1}$. To express the multiplicity m mentioned in the previous paragraph, we use an edge-bicolored graph G, i.e., G is a graph consisting of the vertex set $V_G = \{v_1, v_2, \ldots, v_{\ell+1}\}$ and the set of edges E_G which has the decomposition $E_G = E_G^+ \cup E_G^-$ with $E_G^+ \cap E_G^- = \emptyset$. Then we can define the following map.

Definition 0.1
The map m_G on the braid arrangement $\mathcal{A}_\ell$ is defined by

$$m_G(H_{ij}) := \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E_G^+, \\ -1 & \text{if } \{v_i, v_j\} \in E_G^-, \text{ and} \\ 0 & \text{otherwise}, \end{cases}$$

where $\{v_i, v_j\}$ denotes the undirected edge between v_i and v_j.

Also, we introduce the following notion of edge-bicolored graphs to characterize the freeness.

Definition 0.2
The graph G is bicolor-eliminable with a bicolor-elimination ordering $\nu : V_G \to \{1, 2, \ldots, \ell+1\}$ if ν is bijective, and for every three vertices $v_i, v_j, v_k \in V_G$ with $\nu(v_i), \nu(v_j) < \nu(v_k)$, the induced subgraph $G|_{\{v_i,v_j,v_k\}}$ is neither (1) nor (2) in the following:

(1) For $\sigma \in \{+, -\}$, $\{v_i, v_k\}$ and $\{v_j, v_k\}$ are edges in E_G^σ, and $\{v_i, v_j\} \notin E_G^\sigma$.

(2) For $\sigma \in \{+, -\}$, $\{v_k, v_i\} \in E_G^\sigma$, $\{v_i, v_j\} \in E_G^{-\sigma}$ and $\{v_k, v_j\} \notin E_G$.

For a bicolor-eliminable graph G with a bicolor-elimination ordering ν, $v \in V_G$ and $i \in \{1, 2, \ldots, \ell+1\}$, define the degree $\widetilde{\deg}_i(v)$ by

$$\widetilde{\deg}_i(v) := \deg(v, V_G, E_G^+|_{\nu^{-1}\{1,2,\ldots,i\}}) - \deg(v, V_G, E_G^-|_{\nu^{-1}\{1,2,\ldots,i\}}),$$

where $\deg(w, V_H, E_H) := |\{x \in V_H | \{w, x\} \in E_H\}|$ is the degree of the vertex w in the graph $H = (V_H, E_H)$, and $(V_G, E_G^\sigma|_S)$ with respect to $S \subset V_G$ is the induced subgraph of G whose set of edges consists of $\{\{i, j\} \in E_G | i, j \in S\}$. Furthermore, define $\widetilde{\deg}_i := \widetilde{\deg}_i(\nu^{-1}(i))$ for each i $(1 \le i \le \ell+1)$.

We consider the property of bicolor-eliminable graphs in Sections two and three. Also note that a bicolor-eliminable graph is a generalization of a chordal graph, or a graph which has a vertex elimination order (see Remark 2.3). By using chordal graphs, Stanley classified completely the free and non-free graphic arrangements in [20] (see also [11] or Section one in this article).

What we will do in this article is the multi-version of Stanley's result. In other words, we will classify free multiplicities on the braid arrangement of the form $2k + m_G$ with m_G defined in Definition 0.1 in more general setting. The main result is the following characterization of the freeness in terms of bicolor-eliminable graphs.

Theorem 0.3
Let $\mathcal{A}$ be the braid arrangement in $V^{\ell+1}$, G an edge-bicolored graph and m_G the map in Definition 0.1. Let $k, n_1, \ldots, n_{\ell+1}$ be non-negative integers. Define a multi-braid arrangement $(\mathcal{A}, m) = \mathcal{A}_\ell(n_1, n_2, \ldots, n_{\ell+1})[G]$ by $m(H_{ij}) = 2k + n_i + n_j + m_G(H_{ij})$ and put $N = (\ell+1)k + \sum_{i=1}^{\ell+1} n_i$. Assume that one of the following three conditions is satisfied:

(a) $k > 0$.

(b) $E_G^- = \emptyset$.

(c) $E_G^+ = \emptyset$ *and* $m(H_{ij}) > 0$ *for all* $H_{ij} \in \mathcal{A}$.

Then $\mathcal{A}_\ell(n_1, n_2, \ldots, n_{\ell+1})[G]$ is free with

$$\exp(\mathcal{A}, m) = (0, N + \widetilde{\deg}_2, \ldots, N + \widetilde{\deg}_{\ell+1})$$

if and only if G is bicolor-eliminable.

If we let $n_1 = \cdots = n_{\ell+1} = 0$ for case (b) of Theorem 0.3 then the corresponding arrangement is a graphic arrangement where each hyperplane has multiplicity one. Therefore, Theorem 0.3 is a generalization of Stanley's classification of free graphic arrangements. In Sections two and three we will see that a bicolor-eliminable graph is a generalization of the concept of a chordal graph. Hence, Theorem 0.3 generalizes both aspects of Stanley's work in [20]: the freeness of certain arrangements and combinatorial properties of the corresponding graphs.

The organization of this article is as follows. In Section one we introduce some fundamental results and definitions about multiarrangements and their freeness. In Section two we introduce the theory of bicolor-eliminable graphs, which can be regarded as a generalization of the chordal graph theory from the viewpoint of the characterization of free graphic arrangements due to Stanley. In Section three we quote a characterization of bicolor-eliminable graphs from [14]. In Section four we apply the results in the previous sections to the study of free multiplicities on the braid arrangement, and prove Theorem 0.3. In Section five, we give an application of Theorem 0.3 to a conjecture of Athanasiadis in [10].

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1 Preliminaries

In this section let us review some results and definitions which will be used in this article. Let us begin with those for (multi)arrangements of hyperplanes, for which we refer the reader to [15]. First we introduce some results for the study of free and non-free multiarrangements. Let $(\mathcal{A}, m)$ be a multiarrangement in an ℓ-dimensional vector space and fix $H_0 \in \mathcal{A}$ with $m(H_0) > 0$. Define the *deletion* $(\mathcal{A}', m')$ of $(\mathcal{A}, m)$ with respect to H_0 by $\mathcal{A}' = \mathcal{A}$ and

$$m'(H) = \begin{cases} m(H) & \text{if } H \neq H_0, \\ m(H_0) - 1 & \text{if } H = H_0. \end{cases}$$

Theorem 1.1 ([6], Theorem 0.4)
If $(\mathcal{A}, m)$ and $(\mathcal{A}', m')$ are both free, then there exists a basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A}', m')$ such that $\{\theta_1, \ldots, \theta_{k-1}, \alpha_{H_0}\theta_k, \theta_{k+1}, \ldots, \theta_\ell\}$ is a basis for $D(\mathcal{A}, m)$ for some $k \in \{1, \ldots, \ell\}$.

For $X \in \mathcal{A}'' := \{H' \cap H_0 | H' \in \mathcal{A} \setminus \{H_0\}\}$, define $\mathcal{A}_X := \{H \in \mathcal{A} | X \subset H\}$ and $m_X := m|_{\mathcal{A}_X}$. Since $\mathcal{A}_X$ is essentially a 2-multiarrangement, Theorem 1.1 implies that $(\mathcal{A}_X, m_X)$ is free with a basis $\{\zeta_3, \zeta_4, \ldots, \zeta_\ell, \theta_X, \psi_X\}$, where $\deg(\zeta_i) = 0$, $\theta_X \notin \alpha_{H_0}\mathrm{Der}_{\mathbb{K}}(S)$ and $\psi_X \in \alpha_{H_0}\mathrm{Der}_{\mathbb{K}}(S)$. Then we define the *Euler multiplicity* m^* on $\mathcal{A}''$ by $m^*(X) := \deg(\theta_X)$, and we call $(\mathcal{A}'', m^*)$ the *Euler restriction*. Then the following *Addition-Deletion theorem* holds.

Theorem 1.2 ([6], Theorem 0.8)
Let $(\mathcal{A}, m), (\mathcal{A}', m')$ and $(\mathcal{A}'', m^*)$ be the triple with respect to H_0. Then any two of the following statements imply the third:

(i) $(\mathcal{A}, m)$ *is free with* $\exp(\mathcal{A}, m) = (d_1, \ldots, d_{\ell-1}, d_\ell)$.

(ii) $(\mathcal{A}', m')$ *is free with* $\exp(\mathcal{A}', m') = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.

(iii) $(\mathcal{A}'', m^*)$ *is free with* $\exp(\mathcal{A}'', m^*) = (d_1, \ldots, d_{\ell-1})$.

In particular, if $(\mathcal{A}, m)$ and $(\mathcal{A}', m')$ are both free, then all the statements (i), (ii) and (iii) above hold.

In general, the computation of Euler multiplicities m^* is difficult without using a computer program. However, under some special condition, we can obtain m^* in the following manner:

Proposition 1.3 ([6], Proposition 4.1)
Let $(\mathcal{A}, m)$ be a multiarrangement, $H_0 \in \mathcal{A}$ and $(\mathcal{A}'', m^*)$ the Euler restriction of $(\mathcal{A}, m)$ with respect to H_0. Let $X \in \mathcal{A}''$ and put $m_0 = m(H_0)$. Suppose $k = |\mathcal{A}_X|$ and $m_1 = \max\{m(H) | H \in \mathcal{A}_X \setminus \{H_0\}\}$.

(1) If $k = 2$ then $m^*(X) = m_1$.

(2) If $2m_0 \geq |m_X|$ then $m^*(X) = |m_X| - m_0$.

(3) If $2m_1 \geq |m_X| - 1$ then $m^*(X) = m_1$.

(4) If $|m_X| \leq 2k - 1$ and $m_0 > 1$ then $m^*(X) = k - 1$.

(5) If $|m_X| \leq 2k - 2$ and $m_0 = 1$ then $m^*(X) = |m_X| - k + 1$.

(6) If $m_X \equiv 2$ then $m^*(X) = k$.

(7) If $k = 3$, $2m_0 \leq |m_X|$, and $2m_1 \leq |m_X|$ then $m^*(X) = \left\lfloor \frac{|m_X|}{2} \right\rfloor$.

Also, to show the freeness of some deformations of the Coxeter arrangement, the following theorems by Ziegler in [28] and Yoshinaga in [25] play central roles (see Section five). To introduce these results, let us review some definitions. Let $\mathcal{A}$ be a non-empty hyperplane arrangement and $H_0 \in \mathcal{A}$. The *intersection lattice* $L(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$L(\mathcal{A}) := \{\bigcap_{H \in \mathcal{B}} H | \mathcal{B} \subset \mathcal{A}\}$$

with the reverse inclusion as the partial ordering. For $X \in L(\mathcal{A})$ the subarrangement $\mathcal{A}_X \subset \mathcal{A}$ is defined as the set $\{H \in \mathcal{A} | X \subset H\}$. $\mathcal{A}'$ is the *deletion* of $\mathcal{A}$ with respect to H_0, defined by $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$. Also, $\mathcal{A}''$ is the *restriction* of $\mathcal{A}$ with respect to H_0, defined by $\mathcal{A}'' := \{H' \cap H_0 | H' \in \mathcal{A}'\}$. For each $X \in \mathcal{A}''$ we can associate the *Ziegler multiplicity* m_{H_0}, defined in [28], by $m_{H_0}(X) := |\{H' \in \mathcal{A}' | H' \cap H_0 = X\}|$, and we call $(\mathcal{A}'', m_{H_0})$ the *Ziegler restriction* with respect to H_0.

Theorem 1.4 ([28])
In the above notation, if $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$, then $(\mathcal{A}'', m_{H_0})$ is free with $\exp(\mathcal{A}'', m_{H_0}) = (d_2, \ldots, d_\ell)$.

Theorem 1.5 ([25], Theorem 2.2)
In the above notation, assume that $\ell \geq 4$. Then $\mathcal{A}$ is free if and only if $(\mathcal{A}'', m_{H_0})$ is free and $\mathcal{A}_X$ is free for all $X \in L(\mathcal{A}'') \setminus \{\bigcap_{H \in \mathcal{A}} H\}$.

Next we introduce a criterion to check the non-freeness of multiarrangements, see [5] for the notation and details.

Theorem 1.6 ([5], Corollary 4.6)
If a multiarrangement $(\mathcal{A}, m)$ is free, then $GMP(k) = LMP(k)$ $(1 \leq k \leq \ell)$, where $GMP(k)$ is the k-th global mixed product of $(\mathcal{A}, m)$ and $LMP(k)$ is the k-th local mixed product of $(\mathcal{A}, m)$.

The next proposition is useful to determine the non-freeness of multiarrangements, and the proof is the same as that for simple arrangements, see Theorem 4.37 in [15] for example.

Proposition 1.7 ([2], Lemma 3.8)
Let $(\mathcal{A}, m)$ be a multiarrangement and $X \in L(\mathcal{A})$. If $(\mathcal{A}, m)$ is free, then so is $(\mathcal{A}_X, m_X)$.

Next let us review the theory of a graphic arrangement and chordal graph by Stanley in [20]. First, let us consider a subarrangement $\mathcal{B}$ of the Coxeter arrangement of type A_ℓ. Then $\mathcal{B}$ can be uniquely characterized by using the graph G consisting of the vertex set $V_G = \{1, 2, \ldots, \ell + 1\}$ and the set of non-directed edges E_G in the following manner:

Definition 1.8
For a graph G as above, a graphic arrangement $\mathcal{A}_G$ associated to the graph G is defined by

$$\mathcal{A}_G := \{H_{ij} | \{i, j\} \in E_G\}.$$

It is a natural problem to consider whether we can characterize the freeness of graphic arrangements in terms of the combinatorics of G. For that purpose, let us introduce the following graph.

Definition 1.9
Let G be a graph as above. A subgraph $C \subset G$ is a cycle if C consists of vertices $i_1, \ldots, i_s$ $(s \geq 3)$ and $\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{s-1}, i_s\}, \{i_s, i_1\}$ are edges of C. A chord of a cycle C is an edge $\{i, j\}$ for non-consecutive vertices i, j on the cycle C. A graph G is chordal if every cycle $C \subset G$ with $|C| > 3$ has a chord.

It is known that a graph is chordal if and only if its vertex set admits a vertex elimination order, see [13]. By using chordal graphs, Stanley gave a complete classification of free graphic arrangements as follows:

Theorem 1.10 ([20])
A graphic arrangement $\mathcal{A}_G$ is free if and only if G is chordal.

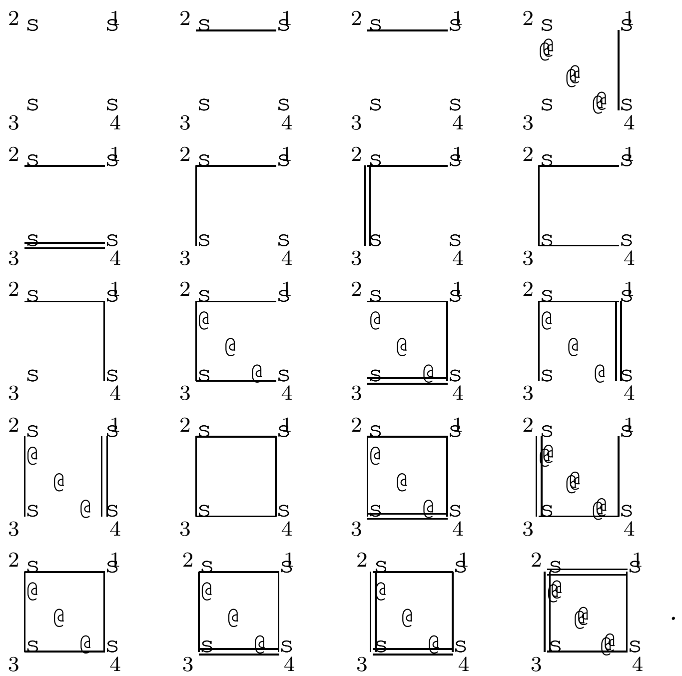
For the rest of this article we give a generalization of Definition 1.9 and Theorem 1.10.

2 Bicolor-eliminable graphs

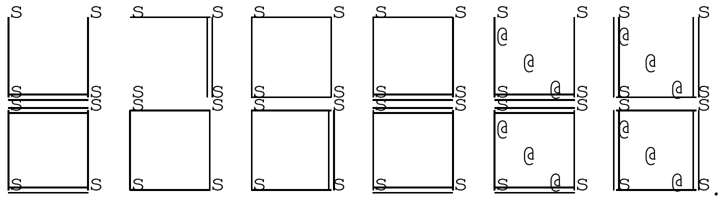
In this section we introduce the theory of bicolor-eliminable graphs and give fundamental properties. This is a generalization of a chordal graph from the viewpoint of its vertex elimination ordering property. Recall the definition of a bicolor-eliminable graph in Definition 0.2 for the multi-braid arrangement. In the rest of this section, we introduce the theory of bicolor-eliminable graphs under the following setting.

Let G be a graph consisting of the vertex set V_G with $|V_G| = \ell$ and the set of edges E_G which has the decomposition $E_G = E_G^+ \cup E_G^-$ such that $E_G^+ \cap E_G^- = \emptyset$. For a subset $S \subset V_G$, $G|_S$ is the induced subgraph of G with $V_{G|_S} = S$. An edge-bicolored graph G is bicolor-eliminable if V_G admits a bicolor-elimination ordering ν. To help understanding, we often consider that the edges in E_G^+ and E_G^- are painted in different colors.

Example 2.1
Let us classify all the bicolor-eliminable and non-bicolor-eliminable graphs with four vertices. Note that, by definition, the property that a graph is bicolor-eliminable is preserved even if we exchange the signs + and −. Now the following graphs are bicolor-eliminable, where the numberings of vertices in the figure signify the corresponding bicolor-elimination ordering (we agree that an edge drawn in a single line belongs to E_G^σ and that in a double line to $E_G^{-\sigma}$ ($\sigma \in \{+, -\}$)):



The following graphs are not bicolor-eliminable:



Definition 2.2
Let ν be a bicolor-elimination ordering on G. We define a k-th bicolor-eliminable filtration of G as a sequence of graphs $G_0, \dots, G_m$ such that

- $G_0 = G|_{\{\nu^{-1}(1),\dots,\nu^{-1}(k-1)\}}$,
- $G_m = G|_{\{\nu^{-1}(1),\dots,\nu^{-1}(k)\}}$,
- *G_i is a subgraph of G_{i+1} with edge-coloring induced by that of G_{i+1},*
- *$|E_{G_{i+1}} \setminus E_{G_i}| = 1$, and*
- *$\nu|_{G_i}$ is a bicolor-elimination ordering on G_i for each i.*

For a bicolor-eliminable graph with ℓ vertices, we define a complete bicolor-eliminable filtration of G as a sequence of graphs $G_0, \dots, G_m$ such that $G_{n_k}, \dots, G_{n_{k+1}}$ is a k-th bicolor-eliminable filtration of G for some $0 = n_1 \leq n_2 \leq \cdots \leq n_{\ell+1} = m$.

Remark 2.3
The definition of a bicolor-eliminable graph with a bicolor-elimination ordering is just a generalization of the vertex elimination order on a graph with one-colored edges. Hence, from the viewpoint of Definition 1.9 and Theorem 1.10, a bicolor-eliminable graph can be regarded as a generalization of a chordal graph. Theorem 3.2 in Section three also supports this generalization.

Let us investigate the properties of bicolor-eliminable graphs. The next proposition follows immediately by definition.

Proposition 2.4
If some induced subgraph of G is not bicolor-eliminable, then G is not bicolor-eliminable either.

Now let us state the main theorem in this section, which will play the key role to characterize free multiplicities on the braid arrangement.

Theorem 2.5
If G is bicolor-eliminable, then G always has a complete bicolor-eliminable filtration.

Roughly speaking, Theorem 2.5 ensures that we can always give an order on edges of a bicolor-eliminable graph which enables Addition-Deletion Theorem 1.2 work well. In the rest of this section we prove Theorem 2.5. For that purpose, we fix the following notation only in the rest of this section. Let G be a bicolor-eliminable graph with ℓ vertices, ν a bicolor-elimination ordering on G, and $l \in V_G$ the vertex $\nu^{-1}(\ell)$.

Lemma 2.6
For $i, j \in V_G$, define the relation $i \prec j$ if $\{i, j\}$ and $\{i, l\}$ are edges of the same color and $\{j, l\}$ is an edge of the other color. Then the relation $\prec$ induces a partial order on $\{i | \{i, l\} \in E_G\}$.

Proof. First, let us show that $i_1 \prec i_2 \prec i_3 \prec i_4$ implies $i_1 \prec i_4$ ($i_a \in V_G$). By symmetry, we may assume that $\{i_1, l\}, \{i_1, i_2\} \in E_G^+$ and $\{i_2, l\} \in E_G^-$. Then $\{i_2, i_3\} \in E_G^-$, $\{i_3, l\}, \{i_3, i_4\} \in E_G^+$ and $\{i_4, l\} \in E_G^-$ by definition of $\prec$. Now if $\{i_1, i_4\} \notin E_G^+$, then Example 2.1 shows that $G|_{\{i_1,i_2,i_3,i_4\}}$ is not bicolor-eliminable, which contradicts Proposition 2.4. Hence $\{i_1, i_4\} \in E_G^+$, and $i_1 \prec i_4$.

Now it suffices to show that there are no vertices $i_1, \ldots, i_n$ ($n \geq 2$) such that $i_1 \prec i_2 \prec \cdots \prec i_n \prec i_1$. If such vertices exist, then repeated use of the argument above implies that $i_1 \prec i_n \prec i_1$ (when n is even) or $i_1 \prec i_2 \prec i_n \prec i_1$ (when n is odd). However, this is impossible by definition of $\prec$. □

Lemma 2.7
Let j be a maximal vertex of the poset $\{i | \{i, l\} \in E_G\}$ defined by $\prec$ in Lemma 2.6 and G' the graph obtained from G by deleting the edge $\{j, l\}$. Then G' is also bicolor-eliminable with the same bicolor-elimination ordering ν.

Proof. By the definition of the bicolor-eliminable graph, it is sufficient to consider the induced subgraph $G'|_{\{i,j,l\}}$ for any i with $\nu(i) < \nu(l)$. The classification of every possible case for $G|_{\{i,j,l\}}$ shows that the induced subgraph $G'|_{\{i,j,l\}}$ does not satisfy the conditions of Definition 0.2 only if $\{i, j\}$ and $\{j, l\}$ are edges of the same color and $\{i, l\}$ is an edge of the other color in $G|_{\{i,j,l\}}$. However, we have assumed that j is a maximal vertex of the poset $\{i | \{i, l\} \in E_G\}$ defined by $\prec$, which completes the proof. □

Proof of Theorem 2.5. Apply Lemma 2.7 repeatedly to edges $\{\{i, l\} \in E_G | \nu(i) < \nu(l)\}$. □

3 Characterization of bicolor-eliminable graphs

In this section we quote a characterization of bicolor-eliminable graphs from [14]. To state it, let us introduce the following two definitions.

Definition 3.1 ([14], Definition 4.4)
Let G be a graph with the set of vertex V_G and two sets of edges E_G^+ and E_G^- as in the previous section, and $\sigma \in \{+, -\}$.

(1) A sequence $(v_1, v_2, \ldots, v_n; \omega)$ $(n \geq 3)$ of vertices in G is a (σ-)mountain if $\{v_i, v_{i+1}\} \in E_G^{-\sigma}$ for $1 \leq i \leq n-1$, $\{\omega, v_i\} \in E_G^{\sigma}$ for $2 \leq i \leq n-1$ and any other pair of vertices is not joined by an edge.

(2) A sequence $(v_1, v_2, \ldots, v_n; \omega_1, \omega_2)$ $(n \geq 2)$ of vertices in G is a (σ-)hill if $\{v_i, v_{i+1}\} \in E_G^{-\sigma}$ for $1 \leq i \leq n-1$, $\{\omega_1, \omega_2\} \in E_G^{\sigma}$, $\{\omega_1, v_i\} \in E_G^{\sigma}$ for $1 \leq i \leq n-1$, $\{\omega_2, v_i\} \in E_G^{\sigma}$ for $2 \leq i \leq n$ and any other pair of vertices is not joined by an edge.

By using chordality, mountains, hills, and Example 2.1, a characterization of bicolor-eliminable graphs is given as follows.

Theorem 3.2 ([14], Theorem 5.1)
Let G be an edge-bicolored graph. Then G is bicolor-eliminable if and only if the following three conditions are satisfied:

(C1) Both graphs (V_G, E_G^+) and (V_G, E_G^-) are chordal.

(C2) Any induced subgraph of G with four vertices is bicolor-eliminable.

(C3) G contains no mountains nor hills.

For details of Theorem 3.2, see [14]. Theorem 3.2 plays the key role for the proof of the "only if" part of Theorem 0.3. Note that, if $E_G^- = \emptyset$, then Theorem 3.2 asserts the well-known equivalence between a chordal graph and a graph with a vertex elimination ordering.

4 Proof of Theorem 0.3

In this section we apply the theory of bicolor-eliminable graphs to prove Theorem 0.3. Since the proof is the same, we only prove the case when the condition (a) in Theorem 0.3 is satisfied.

First, let us prove the "if" part. Let G be a bicolor-eliminable graph with a bicolor-elimination ordering $\nu : V_G \to \{1, 2, \ldots, \ell+1\}$. By an appropriate change of coordinates, we may assume that $\nu(v_i) = i$ for all i. Then let us identify v_i with i for all i in this proof. Hence the order of vertices $V_G = \{1, 2, \ldots, \ell+1\}$ is already a bicolor-elimination ordering. When $E_G = \emptyset$, the theorem is proved in [4], or can be proved by using the argument below with the bicolor-eliminable graph G consisting of $V_G = \{1, 2, \ldots, \ell+1\}$ and $E_G = E_G^+ = \{\{i, j\} | j = 1, \ldots, \ell+1, j \neq i\}$ for a fixed i. We prove the statement by induction on ℓ. When $\ell = 1$ there is nothing to prove. If $\ell = 2$ then the result in [23] completes the proof. Assume that $\ell > 2$. Also, assume that $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[G|_{\{1,2,\ldots,s-1\}}]$ is free with exponents $(0, N + \widetilde{\deg}_2, \ldots, N + \widetilde{\deg}_{s-1}, N, \ldots, N)$ for some s, $2 \leq s \leq \ell+1$. By Theorem 2.5, there exists an s-th filtration $G_0^s, \ldots, G_{f(s)}^s$ of G with $E_{G_{i+1}^s} \setminus E_{G_i^s} = \{\{s, j_i\}\}$ $(j_i < s)$. Consider the Euler restriction $(\mathcal{A}'', m^*)$ of the multiarrangement $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[G_{i+1}^s]$ onto the hyperplane H_{sj_i} $(i = 0, 1, \ldots, f(s)-1)$. Combining Theorem 1.2 and Proposition 1.3 with Definition 0.2 and Theorem 2.5, the lemma below follows immediately.

Lemma 4.1
In the notation above, let $t \in V_G$ with $t < s$. If $(\mathcal{A}'', m^*)$ is the Euler restriction with respect to H_{sj_i}, then

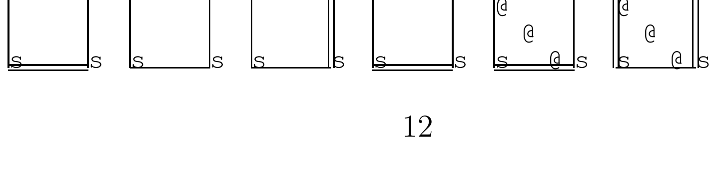
$$m^*(H_{tj_i}) = m^*(H_{ts}) = 3k + n_{j_i} + n_s + n_t + m_G(H_{tj_i}).$$

Then Lemma 4.1 implies that the Euler restriction $(\mathcal{A}'', m^*)$ is equal to the following multiarrangement:

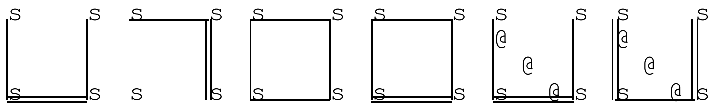
$$\mathcal{A}_{\ell-1}(n_1, \ldots, n_{j_i-1}, n_{j_i} + n_s + k, n_{j_i+1}, \ldots, n_{s-1}, n_{s+1}, \ldots, n_{\ell+1})[G|_{\{1,2,\ldots,s-1\}}].$$

Proposition 2.4 and Theorem 2.5 imply that $G|_{\{1,2,\ldots,s-1\}}$ is also bicolor-eliminable with a bicolor-elimination ordering $\{1, 2, \ldots, s-1\}$. Hence the induction hypothesis shows that $(\mathcal{A}'', m^*)$ is free with exponents $(0, N + \widetilde{\deg}_2, \ldots, N + \widetilde{\deg}_{s-1}, N, \ldots, N)$. Then Addition-Deletion Theorem 1.2 completes the proof of the "if" part.

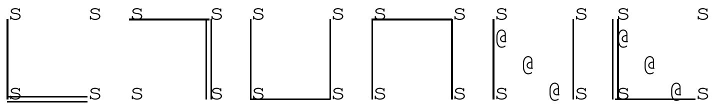
Next we prove the "only if" part. Assume that G is not bicolor-eliminable. Then Theorem 3.2 implies that G does not satisfy the conditions (C1), (C2) or (C3). Also identify v_i with i for all i in this proof. We will prove that $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[G]$ is not free in each of these three cases. To prove it, let us introduce a definition used only in this proof. An edge-bicolored graph G is *free* if the associated multi-braid arrangement $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[G]$ is free. First, assume that G does not satisfy the condition (C2). Then G contains some non-bicolor-eliminable subgraph with four vertices. By Example 2.1, such a graph is one of the following:



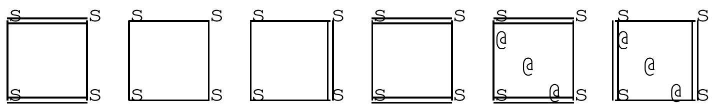
By Proposition 1.7 it suffices to show that these graphs are not free. For that purpose, we use two theorems, i.e., Theorems 1.2 and 1.6. First, prove the non-freeness of the graphs



by using Theorem 1.2. Let us call these graphs of type A. Note that, by deleting an appropriate edge from graphs of type A, we can obtain bicolor-eliminable graphs as follows:



By the proof of the "if" part, these graphs are free. If graphs of type A are also free, then Theorem 1.2 implies that $\exp(\mathcal{A}'', m^*) \subset \exp(\mathcal{A}', m')$ as multisets, which contradicts the results in [23], Proposition 1.3 and what is proved in the "if" part. Hence graphs of type A are not free. Next let us prove the non-freeness of the remaining graphs



by using Theorem 1.6. Let us call these graphs of type B and give a name $B_1, B_2, \ldots, B_6$ to each of these graphs from the left. Assume that graphs of type B are free. Also, assume that a single line edge corresponds to an edge in E_G^+ and a double line edge to that in E_G^-. Let G_i (resp. L_i) denote the 2nd global (resp. local) mixed product of $\mathcal{A}_3(n_1, n_2, n_3, n_4)[B_i]$. Then we can compute these values according to [5] as follows (where $N = \sum_{i=1}^4 n_i$):

$B_1 :\ G_1 \leq 48k^2 + 24kN + 3N^2 < L_1 = 48k^2 + 24kN + 3N^2 + 2.$

$B_2 :\ G_2 \leq 48k^2 + 24kN + 3N^2 + 6N + 24k + 3 < L_2 = 48k^2 + 24kN + 3N^2 + 6N + 24k + 4.$

$B_3 :\ G_3 \leq 48k^2 + 24kN + 3N^2 + 8k + 2N < L_3 = 48k^2 + 24kN + 3N^2 + 8k + 2N + 1.$

$B_4 :\ G_4 \leq 48k^2 + 24kN + 3N^2 + 8k + 2N < L_4 = 48k^2 + 24kN + 3N^2 + 8k + 2N + 2.$

$B_5 :\ G_5 \leq 48k^2 + 24kN + 3N^2 < L_5 = 48k^2 + 24kN + 3N^2 + 1.$

$B_6 :\ G_6 \leq 48k^2 + 24kN + 3N^2 + 4N + 16k + 1 < L_6 = 48k^2 + 24kN + 3N^2 + +4N + 16k + 3.$

Hence Theorem 1.6 implies contradictions, which show that these graphs are not free. Since the same proof as the above is valid when the colors of single and double lines are exchanged, graphs of type B are not free, which shows that every non-bicolor-eliminable graph with four vertices is not free.

Next assume that the condition (C1) is not satisfied. Then there exists a subgraph $C \subset G$ such that $|C| \geq 4$ and $(V_C, E_G^\sigma \cap E_C)$ is a cycle without chords of the color $\sigma \in \{+, -\}$. Because of the symmetry we may assume that $\sigma = +$. Moreover, Proposition 1.7 implies that it is sufficient to show that C or its subgraph is not free. We prove the non-freeness by induction on $\ell \geq 2$. If $\ell = 2$ then there is nothing to prove, so assume that $\ell > 2$. If $|C| = 4$ then Example 2.1 implies that C is not bicolor-eliminable, hence the above arguments imply the non-freeness. Assume that $|C| > 4$. First, assume that there are no chords in $E_C^+ \cup E_C^-$. When $|C| < \ell + 1$, the induction hypothesis completes the proof. So we may assume that $|C| = \ell+1$. We may also assume that $\{\{1, 2\}, \{2, 3\}, \ldots, \{\ell, \ell + 1\}, \{\ell + 1, 1\}\} = E_C^+ = E_C$. Define a subgraph $C' \subset C$ which is obtained from C by deleting the edge $\{\ell + 1, 1\}$. Note that C' is bicolor-eliminable. Then the "if" part of Theorem 0.3 implies that $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[C']$ is free with exponents $(0, N + 1, \ldots, N+1)$. If $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[C]$ is free, then every statement in Theorem 1.2 holds. Let us consider the Euler restriction of $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[C]$ onto $x_{\ell+1} - x_1 = 0$. Then the Euler restriction is equivalent to $\mathcal{A}_{\ell-1}(n_1 + n_{\ell+1} + k, n_2, \ldots, n_\ell)[C'']$, where C'' is a cycle with $V_{C''} = \{1, 2, \ldots, \ell\}$ and $E_{C''} = E_{C''}^+ = \{\{1, 2\}, \{2, 3\}, \ldots, \{\ell - 1, \ell\}, \{\ell, 1\}\}$. If $\ell = 3$, then [23] implies the contradiction on the exponents. If $\ell > 3$ then the induction hypothesis shows that the Euler restriction is not free, which is also a contradiction.

So we may assume that the cycle C contains a chord whose color is $-$. Use the same notation in the above paragraph and assume that the chord is $\{i, j\}$, where i and j are non-consecutive vertices in V_C with $i < j$. Also we may assume that $i \neq 1$ and $j \neq \ell + 1$. Then we obtain two new graphs C_1 and C_2 as induced subgraphs of C with $V_{C_1} = \{1, 2, \ldots, i, j, j + 1, \ldots, \ell + 1\}$ and $V_{C_2} = \{i, i + 1, \ldots, j\}$ respectively. If $|C_1| = 4$ or $|C_2| = 4$, then the previous argument for the non-freeness of non-bicolor-eliminable graphs with four vertices and Example 2.1 complete the proof. If, for example, $|C_1| > 4$, then we may take a subgraph $C_1' \subset C_1$ whose vertices consist of $\{i-1, i, j, j+1\}$. If $E_{C_1'} = \{\{i - 1, i\}, \{i, j\}, \{j, j + 1\}\}$, then C_1' is not bicolor-eliminable with four vertices, hence not free as we have already proved in the above. If there is some other edge in C_1', then the assumption implies that edge has to be colored by $-$. If that edge is $\{i-1, j\}$, then consider the induced subgraph $C_{11} \subset C_1$ whose vertices consist of $\{1, 2, \ldots, i-1, j, j+1, \ldots, \ell+1\}$ and apply the same arguments above. Then finally, we obtain a non-bicolor-eliminable, hence non-free subgraph with four vertices, which completes the proof.

Finally, assume that the condition (C3) is not satisfied. Because the proof is the same, let us assume that G contains a (+)-mountain $C = (v_1, v_2, \ldots, v_s; \omega) \subset G$ ($s \geq 3$). By Proposition 1.7 it suffices to show that C is not free. If $s = 3$, then Example 2.1 implies that C is not bicolor-eliminable. Hence the first argument of the "only if" part of Theorem 0.3 shows the non-freeness. Assume that $s > 3$. Consider the subgraph $C' \subset C$ which is obtained from C by deleting the vertex v_s and the edge $\{v_{s-1}, v_s\} \in E_G^-$. Then C' has a bicolor-elimination ordering whose k-th filtration is given by first adding $\{w, v_{k-1}\}$ and second adding $\{v_{k-2}, v_{k-1}\}$, hence C' is free by the "if" part of Theorem 0.3. If $\mathcal{A}_\ell(n_1, \ldots, n_{v_{\ell+1}})[C]$ is free, then Theorem 1.2 implies that the Euler restriction $(\mathcal{A}'', m^*)$ of $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[C]$ onto $H_{v_{s-1}v_s}$ is also free. However, Proposition 1.3 implies that the Euler restriction $(\mathcal{A}'', m^*)$ corresponds to the graph of the mountain $(v_1, v_2, \ldots, v_{s-1}; \omega)$, hence not free by the induction hypothesis.

When G contains a hill, the same proof as the above can be applied, which completes the proof of Theorem 0.3. $\square$

Since exponents do not depend on a choice of a basis as the multiset, the next corollary follows immediately from Theorem 0.3.

Corollary 4.2
If G is bicolor-eliminable, then $\widetilde{\deg}_1 = 0$ and $(\widetilde{\deg}_1, \widetilde{\deg}_2, \ldots, \widetilde{\deg}_{\ell+1})$ does not depend on a choice of a bicolor-elimination ordering as the multiset.

In [5], a characteristic polynomial $\chi(\mathcal{A}, m, t)$ of multiarrangements is defined and the factorization theorem is proved. In general, the computation of $\chi(\mathcal{A}, m, t)$ is difficult, but if $(\mathcal{A}, m)$ is free, then we can easily compute it by the factorization. So when G is bicolor-eliminable, we can calculate its characteristic polynomial as follows:

Corollary 4.3
Let $(\mathcal{A}, m) = \mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[G]$ be the same as in Theorem 0.3. Define $(\mathcal{A}, \tilde{m})$ by $\tilde{m}(H_{ij}) := 2k + n_i + n_j - m_G(H_{ij})$.

(1) Let $k > 0$. Then $(\mathcal{A}, m)$ is free if and only if $(\mathcal{A}, \tilde{m})$ is free.

(2) If G is bicolor-eliminable, then

$$\chi(\mathcal{A}, m) = t \prod_{i=2}^{\ell+1} (t - N - \widetilde{\deg}_i)$$

and

$$\chi(\mathcal{A}, \tilde{m}) = t \prod_{i=2}^{\ell+1} (t - N + \widetilde{\deg}_i).$$

Corollary 4.3 shows that there exists a duality of exponents of free multibraid arrangements as mentioned in [7].

5 Conjecture of Athanasiadis

In this section we apply the results in previous sections to a conjecture of Athanasiadis in [10]. To state it, let us introduce some notation.

Let us consider an affine arrangement in $V^{\ell+1}$ defined by

$$(5.1) \qquad x_i - x_j = -k - \epsilon(i,j), -k, -(k-1), \ldots, k, k + \epsilon(j,i) \\ (1 \le i < j \le \ell + 1),$$

where $k \in \mathbb{Z}_{\ge 0}$ and $\epsilon(i,j) = 0$ or 1. Note that in this section, we distinguish (i,j) and (j,i) as explained later. Such an arrangement is called a *deformation of the Coxeter arrangement*, and was first investigated systematically by Stanley in [21]. From the viewpoint of the combinatorics and freeness, these arrangements have been extensively studied by Athanasiadis [8], [9], [10], Edelman and Reiner [12], Postnikov and Stanley [16], Yoshinaga [25] and many other authors. The main focus of these authors is on the characteristic polynomial of these arrangements. Because of Terao's factorization theorem, it is important to consider the freeness of these arrangements.

Now let us go back to the deformation (5.1). A useful way to consider this arrangement is introduced by Athanasiadis in [8]. Consider the directed graph G consisting of the vertex set $V_G = \{1, 2, \ldots, \ell + 1\}$ and the set of directed edges $E_G \subset \{(i,j) | 1 \le i,\ j \le \ell + 1\}$. Here the edge (i,j) is the arrow from i to j. If we define

$$\epsilon(i,j) := \begin{cases} 1 & \text{if } (i,j) \in E_G, \\ 0 & \text{if } (i,j) \notin E_G, \end{cases}$$

then every affine arrangement above can be expressed by these directed graphs. For such a graph G let $\mathcal{A}_G$ denote the corresponding arrangement of the form (5.1). In [8], Athanasiadis gave a splitting formula of the characteristic polynomial of $\mathcal{A}_G$ when G satisfies the following two conditions:

(A1) For every triple i, j, h with $i, j < h$, it holds that, if $(i,j) \in E_G$, then $(i,h) \in E_G$ or $(h,j) \in E_G$.

(A2) For every triple i, j, h with $i, j < h$, it holds that, if $(i,h) \in E_G$ and $(h,j) \in E_G$ then $(i,j) \in E_G$.

Athanasiadis also gave the following conjecture.

Conjecture 5.1 ([10], Conjecture 6.6)
Let $k = 0$ in the deformation (5.1). Then the coning $c\mathcal{A}_G$ of $\mathcal{A}_G$ is free if and only if G satisfies conditions (A1) and (A2).

In the rest of this section let us prove that (A1) and (A2) are sufficient conditions in Conjecture 5.1 in more general setting. First, let us prove the following.

Proposition 5.2
Let $H_\infty \in c\mathcal{A}_G$ be the infinity hyperplane of the coning $c\mathcal{A}_G$ of $\mathcal{A}_G$ in (5.1). If G satisfies (A1) and (A2), then the Ziegler restriction $(\mathcal{A}'', m_{H_\infty})$ with respect to H_∞ is of the form $\mathcal{A}_\ell(n_1, \ldots, n_{\ell+1})[G]$ for some $n_1, \ldots, n_{\ell+1}$ and bicolor-eliminable graph G. In particular, it is free.

Proof. Note that the bicolor-eliminability is a local condition. In other words, that can be determined by checking the behavior of edges between every ordered triple of vertices $i, j < h$. Hence the proposition follows immediately by conditions (A1), (A2), the definition of a bicolor-eliminable graph and Theorem 0.3. $\square$

Theorem 5.3
In the deformation (5.1), $c\mathcal{A}_G$ is free if G satisfies (A1) and (A2). In particular, the "if" part of Conjecture 5.1 is true.

Proof. Induction on $\ell \geq 1$. When $\ell = 1$, there is nothing to prove. If $\ell = 2$ then the classification in [1] completes the proof. Assume that $\ell \geq 3$. By Theorem 1.5 and Proposition 5.2, it suffices to show that $(c\mathcal{A}_G)_X$ is free for any $X \in L(c\mathcal{A}_G)$ with $\bigcap_{H \in c\mathcal{A}_G} H \subsetneq X \subset H_\infty$. Again, recall that conditions (A1) and (A2) are local and note that $(c\mathcal{A}_G)_X$ decomposes into the direct product of the empty arrangement and the arrangement $c\mathcal{A}_{G'}$, where G' is some directed graph. In fact, if $X = \{x_{i_1} = x_{i_2} = \cdots = x_{i_s}\} \cap H_\infty$, then G' is the induced subgraph of G with $V_{G'} = \{i_1, \ldots, i_s\}$. Then again the locality of (A1) and (A2) implies that G' also satisfies conditions (A1) and (A2). Since $\mathrm{rank}(c\mathcal{A}_{G'}) < \mathrm{rank}(c\mathcal{A}_G)$, the induction hypothesis implies that $c\mathcal{A}_{G'}$ is free. Hence $(c\mathcal{A}_G)_X$ is also free, which completes the proof. $\square$

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